L06: Model $y = X\beta + e$, $e \sim (0,\, \sigma^2 V)$ under $G\beta = 0$.

1. Concepts

   Consider Model $y = X\beta + e$, $e \sim (0,\, \sigma^2 V)$ under $G\beta = 0$.

   (1) GLSE

$$
\begin{aligned}
& \widehat{\beta} \text{ is GLSE for } \beta \text{ with respect to } \langle\cdot,\cdot\rangle_D \text{ under } G\beta = 0 \\
\overset{def}{\iff}\ & G\widehat{\beta} = 0 \text{ and } \|y - X\widehat{\beta}\|_D^2 \le \|y - X\beta\|_D^2 \text{ for all } \beta \in \mathcal{N}(G) \\
\iff\ & \widehat{\beta} \in \mathcal{N}(G) \text{ and} \\
& \|D^{1/2}y - D^{1/2}X\widehat{\beta}\|^2 \le \|D^{1/2}y - D^{1/2}X\beta\|^2 \text{ for all } \beta \in \mathcal{R}(I - G^+G) \\
\iff\ & \widehat{\beta} \in \mathcal{N}(G) \text{ and } D^{1/2}X\widehat{\beta} = \pi\left(D^{1/2}y \mid \mathcal{R}(D^{1/2}X(I - G^+G))\right) \\
\iff\ & \widehat{\beta} \in \mathcal{N}(G) \text{ and } \widehat{\beta} \in [D^{1/2}X(I - G^+G)]^+D^{1/2}y + \mathcal{N}(X) \\
\iff\ & \widehat{\beta} \in [D^{1/2}X(I - G^+G)]^+D^{1/2}y + \mathcal{N}(X) \cap \mathcal{N}(G)
\end{aligned}
$$

   (2) LUE

$$
\begin{aligned}
Ly \text{ is a LUE for } A\beta \text{ under } G\beta = 0 \ &\overset{def}{\iff}\ E(Ly) = A\beta \text{ for all } \beta \in \mathcal{N}(G) \\
&\iff\ LX(I - G^+G) = A(I - G^+G)
\end{aligned}
$$

   (3) Estimable $A\beta$

$$
\begin{aligned}
A\beta \text{ is estimable under } G\beta = 0 \ &\overset{def}{\iff}\ A\beta \text{ has a LUE under } G\beta = 0 \\
&\iff\ A(I - G^+G) = LX(I - G^+G) \text{ for some } L
\end{aligned}
$$

   (4) BLUE

$$
\begin{aligned}
& By \text{ is BLUE for } A\beta \text{ under } G\beta = 0 \\
\overset{def}{\iff}\ & By \text{ is a LUE for } A\beta \text{ under } G\beta = 0 \text{ and} \\
& \qquad \mathrm{Cov}(By) \le \mathrm{Cov}(Ly) \text{ for all LUE } Ly \text{ under } G\beta = 0 \\
\iff\ & BX(I - G^+G) = A(I - G^+G) \text{ and } LVL' - BVB' \ge 0 \text{ for all } L \text{ satisfying} \\
& \qquad LX(I - G^+G) = A(I - G^+G).
\end{aligned}
$$

2. Results

   (1) LUE

   If $A\beta$ is estimable under $G\beta = 0$, then $\left\{A[D^{1/2}X(I - G^+G)]^+D^{1/2}y :\ D > 0\right\}$ is a subset of the collection of all LUE for $A\beta$ under $G\beta = 0$.

   Proof $A\beta$ is estimable under $G\beta = 0$. So $A(I - G^+G) = LX(I - G^+G)$ for some L.
   For $A[D^{1/2}X(I - G^+G)]^+D^{1/2}y = By$, we need to show that

$$BX(I - G^+G) = A(I - G^+G).$$

   Let $H = D^{1/2}X(I - G^+G)$. Then

$$
\begin{aligned}
BX(I - G^+G) &= [AH^+D^{1/2}][X(I - G^+G)] = AH^+H \\
&= A(I - G^+G)H^+H = LX(I - G^+G)H^+H \\
&= LD^{-1/2}HH^+H = LD^{-1/2}H = LX(I - G^+G).
\end{aligned}
$$

(2) BLUE

If $A\beta$ is estimable under $G\beta = 0$, then $A[V^{-1/2}X(I - G^+G)]^+V^{-1/2}y$ is BLUE for $A\beta$ under $G\beta = 0$.

Proof By (1) $A[V^{-1/2}X(I - G^+G)]^+V^{-1/2}y = By$ is a LUE for $A\beta$ under $G\beta = 0$.

Suppose Ly is also a LUE for $A\beta$ under $G\beta = 0$. Then

$$LX(I - G^+G) = A(I - G^+G) = BX(I - G^+G).$$

We show $LVL' - BVB' \geq 0$ under $G\beta = 0$. Let $H = V^{-1/2}X(I - G^+G)$.

$$\begin{aligned} B &= AH^+V^{-1/2} = A(I - G^+G)H^+V^{-1/2} = LX(I - G^+G)H^+V^{-1/2} \\ &= LV^{1/2}HH^+V^{-1/2}. \end{aligned}$$

So $BVB' = (LV^{1/2})'HH^+(LV^{1/2})'$.

Hence $LVL' = BVB' = (LV^{1/2})(I - HH^+)(LV^{1/2})' \geq 0$.

(3) UE for σ^2

Let $H = V^{-1/2}X(I - G^+G)$. Then $\frac{\|y - X\widehat{\beta}\|^2_{V^{-1}}}{n - \text{rank}[X(I - G^+G)]} = \frac{(V^{-1/2}y)'(I - HH^+)(V^{-1/2}y)}{n - \text{rank}[X(I - G^+G)]}$ is an UE for σ^2 under $G\beta = 0$.

Proof Under $G\beta = 0$, $\beta = (I - G^+G)\gamma$. With $V^{-1/2}y \sim N(H\gamma, \sigma^2 I_n)$,

$$\begin{aligned} E[(V^{-1/2}y)'(I - HH^+)(V^{-1/2}y)] &= (H\gamma)'(I - HH^+)(H\gamma) + \text{tr}[(I - HH^+)\sigma^2 I_n] \\ &= 0 + \sigma^2\{n - \text{rank}[X(I - G^+G)]\}. \end{aligned}$$

So $\frac{(V^{-1/2}y)'(I - HH^+)(V^{-1/2}y)}{n - \text{rank}[X(I - G^+G)]}$ is an UE for σ^2 under $G\beta = 0$.

3. Special cases

(1) $V = I$ and $G = 0$

In $y = X\beta + e$, $e \sim (0, \sigma^2 I_n)$,

estimable $A\beta$ has BLUE AX^+y. σ^2 has UE $\frac{y'(I - XX^+)y}{n - \text{rank}(X)}$.

(2) $V = I$ with $G\beta = 0$

In $y = X\beta + e$, $e \sim (0, \sigma^2 I_n)$, under $G\beta = 0$,

estimable $A\beta$ has BLUE $A[X(I - G^+G)]^+y$ under $G\beta = 0$.

Let $H = X(I - G^+G)$. Then σ^2 has UE $\frac{y'(I - HH^+)y}{n - \text{rank}[X(I - G^+G)]}$ under $G\beta = 0$.

(3) $V = V$ and $G = 0$

In $y = X\beta + e$, $e \sim (0, \sigma^2 V)$,

estimable $A\beta$ has BLUE $A(V^{-1/2}X)V^{-1/x}y$.

Let $H = V^{-1/2}X$. Then σ^2 has UE $\frac{(V^{-1/2}y)'(I - HH^+)(V^{-1/2}y)}{n - \text{rank}(X)}$.