Stat873 **HW01**

1. 1 on p223

   In linear model $y = X\beta + e$, $X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Show that $c_1\beta_1 + c_2\beta_2 + c_3\beta_3$ is estimable if and only if $c_1 = c_2 + c_3$.

   Hint: With $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, show that $c_1 = c_2 + c_3 \Longleftrightarrow c' = LX$ for some L.

   With $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, let $h = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

$$\begin{aligned}
c_1 = c_2 + c_3 \quad &\Longleftrightarrow \quad c_1 - c_2 - c_3 = 0 \Longleftrightarrow h'c = 0 \\
&\Longleftrightarrow \quad c \in \mathcal{N}(h') = \mathcal{N}(hh^+) = \mathcal{R}(I_3 - hh^+) \\
&\Longleftrightarrow \quad c = (I_3 - hh^+)u \text{ for some } u \Longleftrightarrow c' = u'(I_3 - hh^+) \\
&\Longleftrightarrow \quad c' = \tfrac{u'}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \tfrac{u'}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
&\Longleftrightarrow \quad c' = \tfrac{u'}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix} X \Longrightarrow c' = LX \text{ for some } L \Longleftrightarrow c'\beta \text{ is estimable.}
\end{aligned}$$

   On the other hand

$$\begin{aligned}
c'\beta \text{ is estimable} \quad &\Longleftrightarrow \quad c' = LX \text{ for some } L \Longrightarrow c'h = LXh = L0 = 0 \\
&\Longrightarrow \quad h'c = 0 \Longleftrightarrow c_1 = c_2 + c_3.
\end{aligned}$$

2. 2. p223

   In $y = X\beta + e$, $\beta \in R^p$. Let $H = \begin{pmatrix} h'_{(i)} \\ \vdots \\ h'_{(p-1)} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix} \in R^{(p-1)\times p}$, $1_p \in R^p$

   and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} \in R^p$. Show that

$$H\beta \text{ is estimable} \Longleftrightarrow \text{“}1'_p c = 0 \Longrightarrow c'\beta \text{ is estimable”}$$

   Hint: $\mathcal{N}(1'_p)$ has dimension $p - 1$.
   But $h_{(i)} \in \mathcal{N}(1'_p)$ and $h_{(1)}, ..., h_{(p-1)}$ are linearly independent.
   So $[h_{(1)}, ..., h_{(p-1)}]$ is a basis of $\mathcal{N}(1'_p)$. Hence $\mathcal{N}(1'_p) = \mathcal{R}(H')$.

$\Rightarrow$: $1_p' c = 0 \Longrightarrow c \in \mathcal{N}(1_p') = \mathcal{R}(H')$. So $c = H'u$ for some u. Hence $c' = u'H$.
$H\beta$ is estimable. So $H = LX$ for some L.
Hence $c' = u'H = u'LX$. Thus $c'\beta$ is estimable.

$\Leftarrow$: $1_p' h_{(i)} = 0$. So $h_{(i)}'\beta$ is estimable. Thus $h_{(i)}' = u_{(i)}'X$ for some $u_{(i)}$ and $i = 1, .., p-1$.

Hence $H = \begin{pmatrix} u_{(1)}' \\ \vdots \\ u_{(p-1)}' \end{pmatrix} X = LX$ for some L. Therefore $H\beta$ is estimable.