L24 Two-way ANOVA without interactions

1. Two-way ANOVA without interactions using μ

 (1) Two-way ANOVA without interactions using μ
 $y = \mu_{ij} + \epsilon$, $i = 1, .., a$; $j = 1, ..., b$ is two-way ANOVA model. Under the restrictions

$$\mu_{i_1 j_1} + \mu_{i_2 j_2} = \mu_{i_1 j_2} + \mu_{i_2 j_1} \text{ for all } i \text{ and } j$$

 it becomes a two-way ANOVA model without interactions. This model is reduced from two-way ANOVA by the hypothesis H_0 that specifies the restrictions.

 (2) Equivalent forms of the restrictions
 (a) $\mu_{i_1 j_1} + \mu_{i_2 j_2} = \mu_{i_1 j_2} + \mu_{i_2 j_1}$ for all i_1, i_2, j_1 and j_2
 (b) $\mu_{11} + \mu_{ij} = \mu_{1j} + \mu_{i1}$ for all i and all j
 (c) $\mu_{11} + \mu_{ij} = \mu_{1j} + \mu_{i1}$ for all $i \neq 1$ and $j \neq 1$
 are equivalent.

 Proof $(a) \Longrightarrow (b) \Longrightarrow (c)$ are trivial.
 (b)$\Leftarrow$(c): If $i = 1$, then $\mu_{11} + \mu_{ij} = \mu_{11} + \mu_{1j}$ and $\mu_{1j} + \mu_{i1} = \mu_{1j} + \mu_{11}$ are equal.
 If $j = 1$, then $\mu_{11} + \mu_{ij} = \mu_{11} + \mu_{i1}$ and $\mu_{1j} + \mu_{i1} = \mu_{11} + \mu_{i1}$ are equal.
 If $i \neq 1$ and $j \neq 1$, by (c), $\mu_{11} + \mu_{ij} = \mu_{1j} + \mu_{i1}$.
 (a)$\Leftarrow$(b): By (b)

$$\begin{aligned} \mu_{i_1 j_1} + \mu_{i_2 j_2} &= (\mu_{1 j_1} + \mu_{i_1 1} - \mu_{11}) + (\mu_{1 j_2} + \mu_{i_2 1} - \mu_{11}) \\ &= \mu_{1 j_1} + \mu_{1 j_2} + \mu_{i_1 1} + \mu_{i_2 1} - 2\mu_{11} \\ \text{and } \mu_{i_1 j_2} + \mu_{i_2 j_1} &= (\mu_{1 j_2} + \mu_{i_1 1} - \mu_{11}) + (\mu_{1 j_1} + \mu_{i_2 1} - \mu_{11}) \\ &= \mu_{1 j_1} + \mu_{1 j_2} + \mu_{i_1 1} + \mu_{i_2 1} - 2\mu_{11} \end{aligned}$$

 are equal.

 (3) $a + b - 1$ populations
 In two-way ANOVA

$$\begin{aligned} y &= \mu_{ij} + \epsilon = (r_1, .., r_a) M \begin{pmatrix} c_1 \\ \vdots \\ c_b \end{pmatrix} + \epsilon = [(c_1, .., c_b) \otimes (r_1, .., r_a)] \mathrm{vec}(M) + \epsilon \\ &= m\mu + \epsilon \end{aligned}$$

 $\mu = \mathrm{vec}(M)$ has ab free components. The restriction (c) in (2) contains $(a-1)(b-1)$ equations that reduce the number of free components in μ to $ab - (a-1)(b-1) = a + b - 1$. Thus two-way ANOVA without interactions represents $a + b - 1$ populations.

2. Two-way ANOVA without interactions using θ

 (1) The no-interaction hypothesis
 In two-way ANOVA

$$\begin{aligned} y = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon \text{ under } &\textstyle\sum_i \alpha_i = 0,\ \sum_j \beta_j = 0, \\ &\textstyle\sum_i (\alpha\beta)_{ij} = 0 \text{ and } \sum_j (\alpha\beta)_{ij} = 0, \end{aligned}$$

 the no interaction hypothesis

$$\begin{aligned} H_0 : \mu_{i_1 j_1} + \mu_{i_1 j_2} = \mu_{i_1 j_2} + \mu_{i_2 j_1} &\text{ for all } i_1,\ i_2,\ j_1,\ j_2 \\ &\Longleftrightarrow H_0 : (\alpha\beta)_{ij} = 0 \text{ for all } i \text{ and all } j \end{aligned}$$

Proof $\Leftarrow$**:** If $(\alpha\beta)_{ij} = 0$ for all i and j, then
$\mu_{i_1j_1} + \mu_{i_1j_2} = (\mu_{..} + \alpha_{i_1} + \beta_{j_1}) + (\mu_{..} + \alpha_{i_1} + \beta_{j_2})$
and $\mu_{i_1j_2} + \mu_{i_2j_1} = (\mu_{..} + \alpha_{i_1} + \beta_{j_2}) + (\mu_{..} + \alpha_{i_2} + \beta_{j_1})$
are equal for all i_1, i_2, j_1 and j_2.

$\Rightarrow$**:** If $\mu_{i_1j_1} + \mu_{i_1j_2} = \mu_{i_1j_2} + \mu_{i_2j_1}$ for all i_1, i_2, j_1, j_2, by $\mu_{ij} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij}$,

$$\begin{aligned}&(\mu_{..} + \alpha_{i_1} + \beta_{j_1} + (\alpha\beta)_{i_1j_1}) + (\mu_{..} + \alpha_{i_2} + \beta_{j_2} + (\alpha\beta)_{i_2j_2})\\ &= (\mu_{..} + \alpha_{i_1} + \beta_{j_2} + (\alpha\beta) + i_1j_2) + (\mu_{..} + \alpha_{i_2} + \beta_{j_1} + (\alpha\beta)_{i_2j_1}),\end{aligned}$$

i.e., $(\alpha\beta)_{i_1j_1} + (\alpha\beta)_{i_2j_2} = (\alpha\beta)_{i_1j_2} + (\alpha\beta)_{i_2j_1}$ for all i_1, i_2, j_1, j_2.
By the summation of two sides over all $i_1 = 1, ..., a$,

$$0 + a(\alpha\beta)_{i_2j_1} = 0 + a(\alpha\beta)_{i_2j_2} \text{ for all } i_2,\ j_1,\ j_2.$$

Thus $(\alpha\beta)_{ij}$ are equal on all rows over all j with a common value 0.

(2) Two-way ANOVA without interactions
Two-way ANOVA without interactions can be expressed as

$$y = \mu_{..} + \alpha_i + \beta_j + \epsilon = mA_*\theta_* + \epsilon \text{ under } G_*\theta_* = 0$$

where $\theta_* = \begin{pmatrix}\mu_{..}\\ \alpha\\ \beta\end{pmatrix}$, $A_* = (1_b \otimes 1_a,\ 1_b \otimes I_a,\ I_b \otimes 1_a)$ and $G_* = \begin{pmatrix}0 & 1_a' & 0\\ 0 & 0 & 1_b'\end{pmatrix}$

Proof Two-way ANOVA is

$$y = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon = mA\theta + \epsilon \text{ under } G\theta = 0$$

where $\theta = \begin{pmatrix}\mu_{..}\\ \alpha\\ \beta\\ h\end{pmatrix}$, $A = (1_b \otimes 1_a,\ 1_b \otimes I_a,\ I_b \otimes I_a,\ I_b \otimes I_a)$ and $G = \begin{pmatrix}0 & 1_a' & 0 & 0\\ 0 & 0 & 1_b' & 0\\ 0 & 0 & 0 & I_b \otimes 1_a'\\ 0 & 0 & 0 & 1_b' \otimes I_a\end{pmatrix}$.
But under no-interaction H_0, $A\theta = A_*\theta_*$ and $G\theta = 0 \iff G_*\theta_* = 0$ with $\theta_* = \begin{pmatrix}\mu_{..}\\ \alpha\\ \beta\end{pmatrix}$, $A_* = (1_b \otimes 1_a,\ 1_b \otimes I_a,\ I_b \otimes 1_a)$ and $G_* = \begin{pmatrix}0 & 1_a' & 0\\ 0 & 0 & 1_b'\end{pmatrix}$.

(3) $a + b - 1$ populations
In $y = mA_*\theta_* + \epsilon$, θ_* has $1 + a + b$ components. But with two restrictions $\sum_i \alpha_i = 0$ and $\sum_j \beta_j = 0$ in $G_*\theta_* = 0$, the number of free components are $(a + b + 1) - 2 = a + b - 1$. Thus two-way ANOVA without interaction represents $a + b - 1$ populations.

3. Samples

With observed $Y = \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \in R^n$, $\theta_* = \begin{pmatrix}\mu_{..}\\ \alpha\\ \beta\end{pmatrix} \in R^{1+a+b}$, $A^*\theta_*$ loads $\mu_{..} + \alpha_i + \beta_j$ to ab cells. $M = \begin{pmatrix}m_1\\ \vdots\\ m_n\end{pmatrix} \in R^{n\times(ab)}$ where m_k matches y_k to its cell and $MA_*\theta_*$ loads the right $\mu_{..} + \alpha_i + \beta_j$ to y_k.

$$Y = MA_*\theta_* + \epsilon,\ E(\epsilon) = 0 \in R^n, \text{ under } G_*\theta_* = 0.$$