L23 Linear model: Two-way ANOVA

1. Two-way ANOVA

   (1) Populations

   In an experiment Factor A with a levels and factor B with b levels produce ab treatments. Response y to treatments form ab populations.

   $$y = \mu_{ij} + \epsilon \text{ with } E(\epsilon) = 0,\ i = 1, .., a;\ j = 1, .., b.$$

   Let $M = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1b} \\ \vdots & \ddots & \vdots \\ \mu_{a1} & \cdots & \mu_{ab} \end{pmatrix}$ with $\mu = \text{vec}(M)$; $r_i = \begin{cases} 1 & y \text{ is response under level } i \text{ of A} \\ 0 & \text{otherwise} \end{cases}$ is indicator for level i of A and $c_j = \begin{cases} 1 & y \text{ is response under level } j \text{ of B} \\ 0 & \text{0therwise} \end{cases}$ is the indicator for level j of B. So in $m = (c_1, .., c_b) \otimes (r_1, .., r_a)$ the component $r_i c_j$ is the indicator for treatment $(i,\, j)$. Then

   $$y = (r_1, .., r_a) M \begin{pmatrix} c_1 \\ \vdots \\ c_b \end{pmatrix} + \epsilon = [(c_1, .., c_b) \otimes (r_1, .., r_a)]\text{vec}(M) + \epsilon = m\mu + \epsilon$$

   This model gives ab populations.

   (2) Samples

   $y_1, ..., y_n$ is a random sample where y_i is observed when $m = m_i$.

   Let $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in R^n$, $D = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \in R^{n \times ab}$. Then $Y = D\mu + \epsilon$ with $E(\epsilon) = 0 \in R^n$.

   Matrix D matches y_i to its treatment and is called the design matrix.

2. Re-parameterization

   (1) Introducing θ

   With $\mu_{.j} = \frac{\sum_i \mu_{ij}}{a}$, $\mu_{i.} = \frac{\sum_j \mu_{ij}}{b}$, $\mu_{..} = \frac{\sum_i \sum_j \mu_{ij}}{ab}$, let $\alpha_i = \mu_{i.} - \mu_{..}$, $\beta_j = \mu_{.j} - \mu_{..}$ and $(\alpha\beta)_{ij} = \mu_{ij} - \mu_{..} - \alpha_i - \beta_j$. With $H = \left((\alpha\beta)_{ij}\right)_{a \times b}$ and $h = \text{vec}(H)$, introducing

   $$\theta = \begin{pmatrix} \mu_{..} \\ \alpha \\ \beta \\ h \end{pmatrix} \in R^{1+a+b+ab} \text{ where } \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_a \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_b \end{pmatrix} \in R^b.$$

   (2) $\mu = A\theta$ with $A = (1_b \otimes 1_a,\ 1_b \otimes I_a,\ I_b \otimes 1_a,\ I_b \otimes I_a) \in R^{ab \times (1+a+b+ab)}$

   Proof Note that $M = (\mu_{ij})_{a \times b} = \left(\mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij}\right)_{a \times b}$
   $= 1_a \mu_{..} 1_b' + I_a \alpha 1_b' + 1_a \beta' I_b + I_a H I_b.$

   $$\begin{aligned} \text{So } \mu = \text{vec}(M) &= (1_b \otimes 1_a)\mu_{..} + (1_b \otimes I_a)\alpha + (I_b \otimes 1_a)\beta + (I_b \otimes I_a)h \\ &= (1_b \otimes 1_a,\ 1_b \otimes I_a,\ I_b \otimes 1_a, I_b \otimes I_a) \begin{pmatrix} \mu_{..} \\ \alpha \\ \beta \\ h \end{pmatrix} = A\theta. \end{aligned}$$

(3) Restrictions $G\theta = 0$ with $G = \begin{pmatrix} 0 & 1'_a & 0 & 0 \\ 0 & 0 & 1'_b & 0 \\ 0 & 0 & 0 & I_b \otimes 1'_a \\ 0 & 0 & 0 & 1'_b \otimes I_a \end{pmatrix} \in R^{(2+a+b)\times(1+a+b+ab)}$

Proof In θ, $\sum_i \alpha_i = \sum_i (\mu_{i.} - \mu_{..}) = 0$, i.e., $1'_a\alpha = 0$;
$\sum_j \beta_j = \sum_j (\mu_{.j} - \mu_{..}) = 0$, i.e., $1'_b\beta = 0$;
$\sum_i (\alpha\beta)_{ij} = \sum_i (\mu_{ij} - \mu_{..} - \alpha_i - \beta_j) = a\mu_{.j} - a\mu_{..} - 0 - a(\mu_{.j} - \mu_{..}) = 0$. Equivalently $1'_a H = 0$, i.e., $(I_b \otimes 1'_a)h = 0$;
$\sum_j (\alpha\beta)_{ij} = \sum_j (\mu_{ij} - \mu_{..} - \alpha_i - \beta_j) = b\mu_{i.} - b\mu_{..} - b(\mu_{i.} - \mu_{..}) - 0 = 0$. Equivalently $H1_b = 0$, i.e., $(1'_b \otimes I_a)h = 0$.

Thus $0 = \begin{pmatrix} 0 & 1'_a & 0 & 0 \\ 0 & 0 & 1'_b & 0 \\ 0 & 0 & 0 & I_b \otimes 1'_a \\ 0 & 0 & 0 & 1'_b \otimes I_a \end{pmatrix} \begin{pmatrix} \mu_{..} \\ \alpha \\ \beta \\ h \end{pmatrix} = G\theta.$

3. Model with restrictions

(1) Equivalent models
For populations $y = m\mu + \epsilon \Longleftrightarrow y = mA\theta + \epsilon$ under $G\theta = 0$.
For samples $Y = M\mu + \epsilon \Longleftrightarrow Y = MA\theta + \epsilon$ under $G\theta = 0$.

(2) Comments
μ has ab free components. θ has $1 + a + b + ab$ components. Each equation reduces the number free parameters by 1. Among $2 + a + b$ equations in $G\theta = 0$, the last is implied by the previous ones. Consequently, in θ there are $(1 + a + b + ab) - (1 + a + b) = ab$ free components.

Ex: $\theta = B\mu$ with $B = \begin{pmatrix} (1_b \otimes 1_a)^+ \\ [1_b \otimes (I_a - 11^+)]^+ \\ [(I_b - 11^+) \otimes 1_a]^+ \\ [(I_b - 11^+) \otimes (I_a - 11^+)]^+ \end{pmatrix}$.

Proof In $\theta = \begin{pmatrix} \mu_{..} \\ \alpha \\ \beta \\ h \end{pmatrix}$, $\mu_{..} = \frac{\sum_i \sum_j \mu_{ij}}{ab} = \frac{1'_a M 1_b}{ab} = (1_b^+ \otimes 1_a^+)\mu = (1_b \otimes 1_a)^+\mu$.

$\alpha = M\frac{1_b}{b} - 1_a \frac{1'_a M 1_b}{ab} = (I_a - 11^+)M\frac{1_b}{b} = [1_b^+ \otimes (I_a - 11^+)]\mu = [1_b \otimes (I_a - 11^+)]^+\mu$.
$\beta' = \frac{1'_a M}{a} - \frac{1'_a M 1_b}{ab}1'_b = 1_a^+ M(I_b - 11^+)$. So $\beta = [(I_b - 11^+) \otimes 1_a^+]\mu = [(I_b - 11^+) \otimes 1_a]^+\mu$.

$$\begin{aligned} H &= ((\alpha\beta)_{ij})_{a\times b} = (\mu_{ij} - \mu_{..} - \alpha_i - \beta_j)_{a\times b} = M - 1_a\mu_{..}1'_b - \alpha 1'_b - 1_a\beta' \\ &= M - 1_a\frac{1'_a M 1_b}{ab}1'_b - (I_a - 11^+)M\frac{1_b}{b}1'_b - 1_a 1_a^+ M(I_b - 11^+) \\ &= M + 1_a 1_a^+ M 1_b 1_b^+ - M 1_b 1_b^+ - 1_a 1_a^+ M = (I_a - 11^+)M(I_b - 11^+) \end{aligned}$$

So $h = [(I_b - 11^+) \otimes (I_a - 11^+)]\mu = [(I_b - 11^+) \otimes (I_a - 11^+)]^+\mu$.

Therefore $\theta = \begin{pmatrix} (1_b \otimes 1_a)^+ \\ [1_b \otimes (I_a - 11^+)]^+ \\ [(I_b - 11^+) \otimes 1_a]^+ \\ [(I_b - 11^+) \otimes (I_a - 11^+)]^+ \end{pmatrix} \mu = B\mu.$