Stat 872 Exam 2 Oct. 09, 2025

Name:

1. $(A_1, A_2, A_3) \in R^{3\times 3}$ is an orthogonal matrix.

   (1) Find $\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_2^T)]$. (15 points)

   Method I:
   $$\begin{aligned}\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_2^T)] &= \mathrm{tr}[(A_1^T A_1) \otimes (A_2 A_2^T)] \\ &= \mathrm{tr}(A_2 A_2^T) = \mathrm{tr}(A_2^T A_2) = \mathrm{tr}(1) = 1.\end{aligned}$$

   Method II:
   $$\begin{aligned}\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_2^T)] &= \mathrm{tr}[(A_2 \otimes A_1^T)(A_1 \otimes A_2^T)] \\ &= \mathrm{tr}(A_2 A_1^T A_1 A_2^T) = \mathrm{tr}(A_2 A_2^T) \\ &= \mathrm{tr}(A_2^T A_2) = \mathrm{tr}(1) = 1.\end{aligned}$$

   (2) Find $\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_3^T)]$. (15 points)

   Method I:
   $$\begin{aligned}\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_3^T)] &= \mathrm{tr}[(A_1^T A_1) \otimes (A_2 A_3^T)] \\ &= \mathrm{tr}(A_2 A_3^T) = \mathrm{tr}(A_3^T A_2) = \mathrm{tr}(0) = 0.\end{aligned}$$

   Method II:
   $$\begin{aligned}\mathrm{tr}[(A_1^T \otimes A_2)(A_1 \otimes A_3^T)] &= \mathrm{tr}[(A_2 \otimes A_1^T)(A_1 \otimes A_3^T)] \\ &= \mathrm{tr}(A_2 A_1^T A_1 A_3^T) = \mathrm{tr}(A_2 A_3^T) \\ &= \mathrm{tr}(A_3^T A_2) = \mathrm{tr}(0) = 0.\end{aligned}$$

   (3) Find $\mathrm{tr}[(A_1^T \otimes A_3^T \otimes A_2)(A_1 \otimes A_2^T \otimes A_3)]$. (20 points)

   $$\begin{aligned}\mathrm{tr}[(A_1^T \otimes A_3^T \otimes A_2)(A_1 \otimes A_2^T \otimes A_3)] &= \mathrm{tr}[(A_1^T \otimes A_2 \otimes A_3^T)(A_1 \otimes A_2^T \otimes A_3)] \\ &= \mathrm{tr}[(A_1^T A_1) \otimes (A_2 A_2^T) \otimes (A_3^T \otimes A_3)] \\ &= \mathrm{tr}[1 \otimes (A_2 A_2^T) \otimes 1] = \mathrm{tr}(A_2 A_2^T) \\ &= \mathrm{tr}(A_2^T A_2) = \mathrm{tr}(1) = 1.\end{aligned}$$

2. For $A \in R^{m\times m}$, $B \in R^{n\times n}$ and $X \in R^{m\times n}$, let $Y = AX + XB$. Find $\frac{\partial \text{vec}(Y)}{\partial\,[\text{vec}(X)]^T}$. (25 points)

$$\begin{aligned}
\text{vec}(Y) &= \text{vec}(AX + XB) = \text{vec}(AXI_n) + \text{vec}(I_m XB) \\
&= (I_n \otimes A)\,\text{vec}(X) + (B^T \otimes I_m)\,\text{vec}(X) \\
&= [(I_n \otimes A) + (B^T \otimes I_m)]\,\text{vec}(X).
\end{aligned}$$

So, $\frac{\partial\,\text{vec}(Y)}{\partial\,[\text{vec}(X)]^T} = (I_n \otimes A) + (B^T \otimes I_m)$.

3. For $A \in R^{n\times p}$ and $B \in R^{q\times n}$ find $\frac{\partial\,[\text{tr}(AXB)]^2}{\partial\,X}$. (25 points)

$$\begin{aligned}
\frac{\partial\,[\text{tr}(AXB)]^2}{\partial\,X} &= \frac{d\,[\text{tr}(AXB)]^2}{d\,\text{tr}(AXB)}\,\frac{\partial\,\text{tr}(AXB)}{\partial\,X} \\
&= 2\,\text{tr}(AXB)\,\textstyle\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\frac{\partial\,\text{tr}(AXB)}{\partial\,AXB}\right)_{st}\left(\frac{\partial\,(AXB)_{st}}{\partial\,X}\right) \\
&= 2\,\text{tr}(AXB)\,\textstyle\sum_{s=1}^{n}\sum_{t=1}^{n}(I_n)_{st}\,A^T E_{n\times n}(s,t)B^T \\
&= 2\text{tr}(AXB)\,A^T B^T.
\end{aligned}$$