Stat 872 Exam 1 Sept. 11, 2025

Name:

1. $A \in R^{m\times n}$ with rank$(A) = n$. Circle TRUE or FALSE. (30 points)

   (1) The columns of A form a basis for $R(A)$. TRUE✓ FALSE

   (2) In QR-decomposition $A = QR$, the columns of Q form an orthonormal basis for $R(A)$. TRUE✓ FALSE

   (3) rank$(AB) =$ rank(B). TRUE✓ FALSE

   (4) $AA' > 0$. TRUE FALSE✓

   (5) $AA^+ = I$. TRUE FALSE✓

   (6) $(AA')^+ = (A')^+A^+$. TRUE✓ FALSE

2. For $A \in R^{m\times n}$ and non-singular $U \in R^{m\times m}$ define $B = (UA)^+U$. (30 points)

   (1) $ABA = A$. TRUE✓ FALSE

   $ABA = A[(UA)^+U]A = U^{-1}(UA)(UA)^+(UA) = U^{-1}(UA) = A$

   (2) $BAB = B$. TRUE✓ FALSE

   $BAB = [(UA)^+U]A[(UA)^+U] = (UA)^+(UA)(UA)^+U = (UA)^+U = B$

   (3) AB is symmetric. TRUE FALSE✓

   $AB = A[(UA)^+U]$ may not be symmetric

   (4) BA is symmetric TRUE✓ FALSE

   $BA = [(UA)^+U]A = (UA)^+(UA)$ is symmetric

   (5) If U is non-singular, symmetric and idempotent, then $B = A^+$. TRUE✓ FALSE

   $U' = U \Longrightarrow U = P\Lambda P'$ by EVD; U is non-singular $\Longrightarrow \Lambda$ is non-singular
   $U^2 = U \Longrightarrow P\Lambda P'P\Lambda P' = P\Lambda P' \Longrightarrow \Lambda^2 = \Lambda \Longrightarrow \Lambda = I \Longrightarrow U = PIP' = I$.
   Thus $B = (UA)^+U = (IA)^+I = A^+$.

   (6) If U is orthogonal, then $B = A^+$ TRUE✓ FALSE

   $B = (UA)^+U = A^+U^+U = A^+U'U = A^+I = A^+$

3. $A = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

(1) Are the rows of A linearly independent? orthogonal? orthonormal? (5 points)

The rows of A are linearly independent, orthogonal, but not orthonormal.

(2) Find A^+. (10 points)

$$A^+ = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^+ = 2\left(\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}^+, \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^+\right) = 2\begin{pmatrix} 0 & 1/2 \\ -1 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & 0 \\ 0 & -1 \end{pmatrix}$$

4. $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(1) Find x^+. (5 points)

$x^+ = \frac{x'}{x'x} = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$

(2) Find A^+. (5 points)

$A^+ = A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

(3) For $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ find B^+. (15 points)

$$B^+ = \begin{pmatrix} 0 & A \\ x & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & x^+ \\ A^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$