Stat871 **HW10**

1. $X_1, ..., X_n$ is a random sample from $N(\mu, \sigma^2)$. Re-parameterize $\theta = \sigma^2$ and $\tau = \frac{\mu}{\sigma^2}$.

   (1) Write $f(x_1, ..., x_n;\, \theta,\, \tau)$ as $\exp\left[p(\theta,\, \tau) + q(x_1, ..., x_n) + r(\theta)T(x_1, .., x_n) + \tau S(x_1, ..., x_n)\right]$. Identify $p(\theta,\, \tau)$, $q(x_1, ..., x_n)$, $r(\theta)$, $T(x_1, .., x_n)$ and $S(x_1, ..., x_n)$.

$$\begin{aligned} f(x_1, .., x_n;\, \theta,\, \tau) &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{\sum_i (x_i - \mu)^2}{-2\sigma^2}\right] \\ &= \exp\left[-\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum_i x_i^2}{2\sigma^2} - \frac{n\mu^2}{2\sigma^2} + \frac{\mu \sum_i x_i}{\sigma^2}\right] \\ &= \exp\left[-\frac{n}{2}\ln(2\pi\theta) - \frac{n}{2}\theta\tau^2 + \frac{1}{-2\theta}\sum_i x_i^2 + \tau \sum_i x_i\right] \\ &= \exp\left[p(\theta,\, \tau) + q(x_1, ..., x_n) + r(\theta)T(x_1, .., x_n) + \tau S(x_1, .., x_n)\right] \end{aligned}$$

where $p(\theta,\, \tau) = -\frac{n}{2}\ln(2\pi\theta) - \frac{n}{2}\theta\tau^2$, $q(x_1, .., x_n) = 0$
$r(\theta) = \frac{1}{-2\theta}$, $T(x_1, .., x_n) = \sum_{i=1}^n x_i^2$ and $S(x_1, .., x_n) = \sum_{i=1}^n x_i$.

   (2) Identify a sufficient statistic for θ and a sufficient statistic for τ.

   By factorization theorem, $T(X_1, .., X_n)$ is sufficient for θ, and $S(X_1, .., X_n)$ is sufficient for τ.

   (3) Show that with respect to θ, the likelihood function has monotone ratio in $T(X_1, .., X_n)$.

   $\theta_1 < \theta_2 \Longrightarrow \Lambda = \frac{f(x_1, .., x_n;\, \theta_2,\, \tau)}{f(x_1, .., x_n;\, \theta_1,\, \tau)} = e^{[p(\theta_2,\, \tau) - p(\theta_1,\, \tau)]} \cdot e^{[r(\theta_2) - r(\theta_1)]T(x_1, .., x_n)}$
   With increasing function of $r(\theta) = \frac{1}{-2\theta}$ on $\theta \in (0, \infty)$, the ratio Λ is an increasing function of $T(x_1, ..., x_n)$.

2. For conditional α-level UMP test on $H_0 : \theta \le \theta_0$ versus $H_a : \theta > \theta_0$ with θ in 1, the conditional pdf of T given S, $f_{T|S}(\cdot) = \frac{f_{(T,\, S)}(t,\, s;\, \theta,\, \tau)}{\int_t f_{(T,\, S)}(t,\, s;\, \theta,\, \tau)\, dt}$ is needed.
   Express $f_{(T,\, S)}(t,\, s;\, \theta,\, \tau)$ via $f_{N(\mu,\, \sigma^2/n)}(\cdot)$, the pdf of $N\left(\mu,\, \frac{\sigma^2}{n}\right)$, and $f_{\chi^2(n-1)}(\cdot)$, the pdf of $\chi^2(n-1)$.
   Hint: $\overline{X} = \frac{\sum X_i}{n} \sim N(\mu,\, \sigma^2/n)$ and $\frac{\sum_i X_i^2 - n\overline{X}^2}{\sigma^2} \sim \chi^2(n-1)$ are independent.

   $M = \overline{X} = \frac{S}{n} \sim N(\mu,\, \sigma^2/n) = N(\theta\tau,\, \tau/n)$ and $V = \frac{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}{\sigma^2} = \frac{T - \frac{1}{n}S^2}{\theta} \sim \chi^2(n-1)$ are independent. So the joint pdf for $(M,\, V)$ is

$$f_{(M,\, V)}(m,\, v;\, \theta,\, \tau) = f_{N(\theta\tau,\, \theta/n)}(m) \cdot f_{\chi^2(n-1)}(v).$$

   But abs $\left|\frac{\partial\,(m, v)}{\partial\,(t, s)}\right| = \text{abs}\begin{vmatrix} m'_t & m'_s \\ v'_t & v'_s \end{vmatrix} = \text{abs}\begin{vmatrix} 0 & \frac{1}{n} \\ \frac{1}{\sigma^2} & \frac{2s}{n\sigma^2} \end{vmatrix} = \frac{1}{n\sigma^2} = \frac{1}{n\theta}$. Thus the joint pdf for $(T,\, S)$ is

$$f_{(T,\, S)}(t,\, s;\, \theta,\, \tau) = f_{(N(\theta\tau,\, \theta/n)}\left(\frac{s}{n}\right) \cdot f_{\chi^2(n-1)}\left([t - \frac{s^2}{n}]/\theta\right) \cdot \frac{1}{n\theta}.$$