Stat871 **HW09**

1. For $H_0: \theta \le 0$ versus $H_a: \theta > 0$
where θ is in the pdf of a logistic distribution $f(x;\theta) = \frac{e^{-(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$, $-\infty < x < \infty$.
based on a sample $X_1, ..., X_n$ from X with this logistic distribution,

$$\phi(U) = \begin{cases} 1 & U > c \\ 0 & U \le c \end{cases} \quad \text{with } E_0[\phi(U)] = \alpha$$

is a locally most powerful α-level test. Find U.

$f(x;\theta) = \frac{e^{-(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$ $\quad \ln f(x;\theta) = -(x-\theta) - 2\ln[1+e^{-(x-\theta)}]$
$[\ln f(x;\theta)]'_\theta = 1 - 2\frac{e^{-(x-\theta)}}{1+e^{-(x-\theta)}} = \frac{1-e^{-(x-\theta)}}{1+e^{-(x-\theta)}}$ $\quad [\ln f(x;0)]'_\theta = \frac{1-e^{-x}}{1+e^{-x}}$. $\quad$ So $U = \sum_{i=1}^n \frac{1-e^{-X_i}}{1+e^{-X_i}}$.

2. $g(x) = f(x) - k_1 f_1(x) - k_2 f_2(x)$ where $k_1 > 0$ and $k_2 > 0$. Let

$$\phi(x) = \begin{cases} 1 & g(x) > 0 \\ r & g(x) = 0 \\ 0 & g(x) < 0 \end{cases} \quad \text{with} \int \phi(x) f_1(x)dx = c_1 \text{ and } \int \phi(x) f_2(x)dx = c_2.$$

Suppose $0 \le \psi(x) \le 1$, $\int \psi(x) f_1(x)dx \le c_1$ and $\int \psi(x) f_2(x)dx \le c_2$. Show that if $\int \psi(x) f(x)dx = \int \phi(x) f(x)dx$, then $\int \psi(x) f_i(x)dx = \int \phi(x) f_i(x)dx$ for all $i = 1, 2$.
Hint: Let $h(x) = \phi(x) - \psi(x)$ and examine
$\int h(x)f(x)dx = \int h(x)g(x)dx + k_1 \int h(x)f_1(x)dx + k_2 \int h(x)f_2(x)dx$

Let $\quad h(x) = \phi(x) - \psi(x)$. $\quad$ Then $\int h(x)f(x)dx = \int \phi(x)f(x)dx - \int \psi(x)f(x)dx = 0$.
In $\quad 0 = \int h(x)f(x)dx = \int h(x)g(x)dx + k_1 \int h(x)f_1(x)dx + k_2 \int h(x)f_2(x)dx$,

$\int h(x)g(x)dx \ge 0$ since

on $[x: g(x) > 0]$, $h(x) = 1 - \psi(x) \ge 0$ such that $h(x)g(x) \ge 0$;
on $[x: h(x) = 0]$, $h(x)g(x) = 0$ such that $h(x)g(x) = 0$;
on $[x: g(x) < 0]$, $h(x) = -\psi(x) \le 0$ such that $h(x)g(x) \ge 0$.

$k_i \int h(x)f_i(x)dx = k_i[\int \phi(x)f_i(x)dx - \int \psi(x)f_i(x)dx] = k_i[c_i - \int \psi(x)f_i(x)dx] \ge 0$, $i = 1, 2$.

Hence $\quad \int h(x)g(x)dx = 0$, $k_1 \int h(x)f_1(x)dx = 0$ and $k_2 \int h(x)f_2(x)dx = 0$.

With $k_i > 0$,

$$k_i \int h(x)f_i(x)dx = 0 \Longrightarrow \int h(x)f_i(x)dx = 0 \Longrightarrow \int \psi(x)f_i(x)dx = \int \phi(x)f_i(x)dx$$

for all $i = 1, 2$.