Stat871 **HW08**

1. Let $X_1, ..., X_n$ be a random sample from Poisson(λ).

   (1) Find $T(X)$ such that the joint pmf of sample is $\exp[p(\lambda) + q(X) + \eta(\lambda)T(X)]$ where $\eta = \eta(\lambda)$ is a 1-1 mapping.

   The joint pmf of a random sample $X_1, ..., X_n$ from Poisson(λ) is

   $$f(x;\,\lambda) = \prod_i \frac{\lambda^{x_i}}{x_i!}e^{-\lambda} = \frac{\lambda^{\sum x_i}}{x_1!\cdots x_n!}e^{-n\lambda} = \exp\left[-n\lambda - \ln(x_1!\cdots x_n!) + (\ln\lambda)\sum X_i\right]$$

   where $\eta = \ln\lambda$ is a 1-1 mapping. So $T(X) = \sum X_i$.

   (2) Point out the distribution of $T(X)$.

   $T(X) = \sum X_i \sim$ Poisson($n\lambda$).

   (3) For $H_0 : \lambda = \lambda_0$ versus $H_a : \lambda \neq \lambda_0$ find a general form of α-level unbiased UMP test.

   $$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_1 & T = c_1 \\ r_2 & T = c_2 \\ 0 & c_1 < T < c_2 \end{cases}$$

   with $E_{\lambda_0}[\phi(T)] = \alpha$ and $E_{\lambda_0}[T\phi(T)] = \alpha E_{\lambda_0}(T)$ is UMP α-level similar unbiased test.

2. Consider UMP in (3) of 1 with $n = 10$, $\lambda_0 = 0.1$ and $\alpha = 0.05$.

   (1) By the first condition on $\phi(T)$, determine the value of c_1 and the range of c_2.

   With $T \sim$ Poisson(1), by Condition 1 $\alpha = E_{\lambda_0}[\phi(T)]$, i.e.,

   $$0.05 = P(T < c_1) + P(T > c_2) + r_1 P(T = c_1) + r_2 P(T = c_2).$$

   So $P(T < c_1) \leq 0.05$. But $P(T = 0) > 0.05$. Thus $c_1 \leq 0 \Longrightarrow c_1 = 0$.
   Now $P(T > c_2) \leq 0.05$. But $P(T > 2) > 0.05$ and $P(T > 3) < 0.05$. Thus $c_2 \geq 3$.

   (2) If $c_1 = 0$ and $c_2 = 4$, find r_1 and r_2.

   With $c_1 = 0$ and $c_2 = 4$, the first condition becomes,
   $0.05 = 0 + 0.00366 + r_1 P(\text{Poisson}(1) = 0) + r_2 P(\text{Poisson}(1) = 4)$, i.e.,

   $$0.36788 r_1 + 0.01533 r_2 = 0.04634.$$

The second equation $\alpha E_{0.1}(T) = E_{0.1}[T\phi(T)]$ becomes

$$0.05 = \sum_{k=5}^{\infty} kP(T=k) + 4r_2 P(T=4) = 0.01898 + 0.06132r_2$$

So $r_1 = 0.10488$, $r_2 = 0.50587$.