Stat871 HW07

1. $X1, ..., X_n$ is a random sample from $N(\mu,\, 1^2)$.

   (1) Find statistic $T(X)$ such that the pdf of sample X is $f(X;\, \mu) = \exp\left[p(\mu) + q(X) + \mu T(X)\right]$.

$$
\begin{aligned}
f(X;\, \mu) &= \tfrac{1}{(2\pi)^{n/2}} \exp\left[\tfrac{\sum_{i=1}^n (X_i-\mu)^2}{-2}\right] = \tfrac{1}{(2\pi)^{n/2}} \exp\left[-\tfrac{n\mu^2}{2} - \tfrac{\sum X_i^2}{2} + \mu \sum X_i\right] \\
&= \exp\left[\left(-\tfrac{n}{2}\ln 2\pi - \tfrac{n\mu^2}{2}\right) + \left(\tfrac{\sum X_i^2}{-2}\right) + \mu \sum X_i\right].
\end{aligned}
$$

   Thus $T(X) = \sum X_i$.

   (2) Find the distribution of $T(X)$.

   $X_1, ..., X_n$ are iid $N(\mu,\, 1)$. Hence $T(X) = \sum X_i \sim N(n\mu,\, n) = N(n\mu,\, (\sqrt{n})^2)$.

2. Sample X has pdf/pmf $f(X;\, \theta) = \exp\left[p(\theta) + q(X) + \theta T(X)\right]$ such that the sufficient statistic $T(X)$ has pdf/pmf $f(t;\, \theta) = a(\theta)b(t)e^{\theta t}$.

   Let $g_1(t) = f(t;\, \theta_0)$ and $g_2(t) = tf(t;\, \theta_0)$. Then the system of two equations (the two conditions for UMP for two-sided alternative hypothesis) $\left\{\begin{array}{lcl} \int_t \phi(t)f(t;\, \theta_0)\, dt &=& \alpha \\ \int_t \phi(t)[f(t;\, \theta_0)]'_\theta\, dt &=& 0 \end{array}\right.$ is equivalent to $\left\{\begin{array}{l} \int_t \phi(t)g_1(t)\, dt = \alpha \\ \int_t \phi(t)g_2(t)\, dt = c \end{array}\right.$. Find c.

$$
\begin{aligned}
0 &= \int_t \phi(t)[f(t;\, \theta_0)]'_\theta\, dt = \int_t \phi(t)[a(\theta_0)b(t)e^{\theta_0 t}]'_\theta\, dt \\
&= \int_t \phi(t)[a'_\theta(\theta_0)b(t)e^{\theta_0 t} + a(\theta_0)b(t)te^{\theta_0 t}]\, dt = \tfrac{a'_\theta(\theta_0)}{a(\theta_0)} \int_t \phi(t)f(t;\, \theta_0)\, dt + \int_t \phi(t)g_2(t)\, dt \\
&= \tfrac{a'_\theta(\theta_0)}{a(\theta_0)}\alpha + c.
\end{aligned}
$$

   Thus $c = -\tfrac{a'_\theta(\theta_0)}{a(\theta_0)}\alpha$.