Stat871 HW06

1. For Bernoulli distribution with mean p, find a statistic $T(X)$ such that the likelihood ratio is monotone increasing function of $T(X)$.

   Beroulli distribution with mean p has pmf $f(x;\, p) = p^x(1-p)^{1-x},\ x = 0,\ 1$
   and likelihood function $L(p) = \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i} = \left(\frac{p}{1-p}\right)^{\sum x_i}(1-p)^n$.
   For $p_1 < p_2$

$$\Lambda = \frac{L(p_2)}{L(p_1)} = \frac{\left(\frac{p_2}{1-p_2}\right)^{\sum x_i}(1-p_2)^n}{\left(\frac{p_1}{1-p_1}\right)^{\sum x_i}(1-p_1)^n} = \left(\frac{1-p_2}{1-p_1}\right)^n\left(\frac{p_2-p_1p_2}{p_1-p_1p_2}\right)^{\sum x_i}$$

   $p_1 < p_2 \Longrightarrow p_1 - p_1p_2 < p_2 - p_1p_2 \Longrightarrow \frac{p_2-p_1p_2}{p_1-p_1p_2} > 1$.
   So the likelihood ratio is a monotone increasing function of $T(X) = \sum_{i=1}^n X_i$.

2. Identify the distribution of $T(X)$ in 1.

   $X_1,\, X_2, ..., X_n$ are iid Bernoulli$(p) \Longrightarrow T(X)\sum_{i=1}^n X_i \sim B(n,\, p)$.
   Thus $T(X)$ in 1 has Binomial distribution with n and p.

3. Find an UMP test on $H_0 : p = 0.3$ versus $H_a : p > 0.3$ at the level $\alpha = 0.05$ with a sample of size 9.

$$\phi(X) = \begin{cases} 1 & T(X) > k \\ r & T(X) = k \\ 0 & T(X) < k \end{cases}$$

   $0.05 = E_{0.3}[\phi(X)] = P(B(9,\, 0.3) > k) + r \cdot P(B(9,\, 0.3) = k.$

   So $r = \frac{0.05-P(B(9,0.3)>k)}{P(B(9,0.3)=k)} = \frac{0.05-P(B(9,0.3)>5)}{P(B(9,0.3)=5)} = \frac{0.05-0.02529}{0.07351} = 0.33614$

   Thus $\phi(X) = \begin{cases} 1 & \sum_{i=1}^9 X_i > 5 \\ 0.33614 & \sum_{i=1}^9 X_i = 5 \\ 0 & \sum_{i=1}^9 X_i < 5 \end{cases}$ gives an UMP test at $\alpha = 0.05$

4. Find an UMP test on $H_0 : p = 0.8$ versus $H_a : p < 0.8$ at the level $\alpha = 0.05$ with a sample of size 9.

$$\phi(X) = \begin{cases} 1 & T(X) < k \\ r & T(X) = k \\ 0 & T(X) > k \end{cases}$$

$0.05 = E_{0.8}[\phi(X)] = P(B(9,\, 0.8) < k) + r \cdot P(B(9,\, 0.8) = k.$

So $r = \frac{0.05 - P(B(9,0.8)<k)}{P(B(9,0.8)=k)} = \frac{0.05 - P(B(9,0.8)<>5)}{P(B(9,0.8)=5)} = \frac{0.05-0.01958}{0.06606} = 0.46049$

Thus $\phi(X) = \begin{cases} 1 & \sum_{i=1}^{9} X_i < 5 \\ 0.46049 & \sum_{i=1}^{9} X_i = 5 \\ 0 & \sum_{i=1}^{9} X_i > 5 \end{cases}$ gives an UMP test at $\alpha = 0.05$