Stat871 HW05

1. For $H_0: \lambda = 0.4$ versus $H_a: \lambda = 0.8$ where λ is the mean of a Poisson population. With a sample of size 10 find most powerful test at level 0.05.
Use link on `www.math.wichita.edu/~xhu/` to access a Poisson distribution calculator.

$\Lambda = \frac{f(X_1,..,X_{10};0.8)}{f(X_1,..,X_{10};0.4)} = \frac{\frac{1}{X_1!\cdots X_{10}!}(0.8)^{X_1+\cdots+X_{10}}e^{-8}}{\frac{1}{X_1!\cdots X_{10}!}(0.4)^{X_1+\cdots+X_{10}}e^{-4}} = 2^{X_1+\cdots+X_{10}}\,e^{-4}$
is an increasing function of $T = X_1 + \cdots + X_{10} \sim \text{Poisson}(10\lambda)$.

Let $$\phi(X) = \begin{cases} 1 & T > c \\ r & T = c \\ 0 & T < c \end{cases} \quad \text{with}$$

$$\begin{aligned} 0.05 &= E_{0.4}[\phi(X)] = P_{0.4}(T > c) + r \cdot P_{0.4}(T = C) \\ &= P(\text{Poisson}(4) > c) + r \cdot P(\text{Poisson}(4) = c) \\ &= P(\text{Poisson}(4) > 8) + r \cdot P(\text{Poisson}(4) = 8) = 0.02136 + r \cdot 0.02977 \\ \text{So, } r &= 0.96204 \end{aligned}$$

Thus $\phi(X) = \begin{cases} 1 & X_1 + \cdots + X_{10} > 8 \\ 0.96204 & X_1 + \cdots + X_{10} = 8 \\ 0 & X_1 + \cdots + X_{10} < 8 \end{cases}$ is MP test at level 0.05.

2. For $H_0: \mu = 8$ versus $H_0: \mu = 4$ where μ is from a population $N(\mu, 10^2)$. With a sample of size 40 find the most powerful test at level 0.05.

$\Lambda = \frac{f(X_1,..,X_{40};4)}{f(X_1,..,X_{40};8)} = \exp\left[\frac{8}{5}(6 - \overline{X}_{40})\right]$ is a decreasing function of $\overline{X}_{40} \sim N(\mu, 1.5811^2)$.

Let $$\phi(X) = \begin{cases} 1 & \overline{X}_{40} \le c \\ 1 & \overline{X}_{40} > c \end{cases} \quad \text{with}$$

$$0.05 = P_8(\overline{X}_{40} \le c) = P(N(8, 1.5811^2) \le c) = P(Z < \tfrac{c-8}{1.5811})$$
$$\Longrightarrow \tfrac{c-8}{1.5811} = -1.645 \Longrightarrow c = 5.3991.$$

So $\phi(X) = \begin{cases} 1 & \overline{X}_{40} \le 5.3991 \\ 0 & \overline{X}_{40} > 5.3991 \end{cases}$. is the most powerful test at level 0.05.

3. $\phi(X)$ is the most powerful test at level α for $H_0: \theta = \theta_0$ versus $H_a: \theta = \theta_1$ by Neyman-Pearson lemma. Show that $\phi(X)$ is unbiased.
Hint: Need to show $E_{\theta_0}[\phi(X)] \le E_{\theta_1}[\phi(X)]$. Let $\psi(X) \equiv \alpha$.

We need to show $E_{\theta_0}[\phi(X)] \le E_{\theta_1}[\phi(X)]$.
Let $\psi(X) \equiv \alpha$. Then $E_\theta[\psi(X)] = \alpha$ for all $\theta = \theta_0, \theta_1$.
$\psi(X)$ is α-level test, but $\phi(X)$ is MP α-level test. So $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.
But $E_{\theta_1}[\psi(X)] = \alpha = E_{\theta_0}[\phi(X)]$. So $E_{\theta_0}[\phi(X)] \le E_{\theta_1}[\phi(X)]$.