Stat871 HW03

1. Suppose $E_\theta(\widehat{\theta}_n) = \theta$ and $\text{Cov}_\theta(\sqrt{n}\,\widehat{\theta}_n) \longrightarrow I^{-1}(\theta)$. Show that $\widehat{\theta}_n$ is an asymptotically efficient estimator for θ.

   By the definition of asymptotically efficient estimators, we need to show $\widehat{\theta}_n \xrightarrow{p} \theta$.

   By Chebyshev inequality and $E_\theta(\widehat{\theta}_n) = \theta$,

$$
\begin{aligned}
\forall\, \epsilon > 0,\ P(\|\widehat{\theta}_n - \theta\| > \epsilon) &= P(\|\widehat{\theta}_n - \theta\|^2 > \epsilon^2) \le \frac{E_\theta\big(\|\widehat{\theta}_n-\theta\|^2\big)}{\epsilon^2} \\
&= \frac{E_\theta\big\{\text{tr}\big[(\widehat{\theta}_n-\theta)'(\widehat{\theta}_n-\theta)\big]\big\}}{\epsilon^2} = \frac{E_\theta\big\{\text{tr}\big[(\widehat{\theta}_n-\theta)(\widehat{\theta}_n-\theta)'\big]\big\}}{\epsilon^2} \\
&= \frac{\text{tr}\big\{E_\theta\big[(\widehat{\theta}_n-\theta)(\widehat{\theta}_n-\theta)'\big]\big\}}{\epsilon^2} = \frac{\text{tr}\big[\text{Cov}_\theta(\widehat{\theta}_n)\big]}{\epsilon^2} \\
&= \frac{\text{tr}\big[n\cdot\text{Cov}_\theta(\widehat{\theta}_n)\big]}{n\epsilon^2} = \frac{\text{tr}\big[\text{Cov}_\theta(\sqrt{n}\,\widehat{\theta}_n)\big]}{n\epsilon^2}.
\end{aligned}
$$

   But $\text{tr}[\text{Cov}_\theta(\sqrt{n}\,\widehat{\theta}_n)] \longrightarrow \text{tr}[I^{-1}(\theta)]$ and $n\epsilon^2 \longrightarrow \infty$. So $\frac{\text{tr}[\text{Cov}_\theta(\sqrt{n}\,\widehat{\theta}_n)]}{n\epsilon^2} \longrightarrow 0$.

   Thus $\widehat{\theta}_n \xrightarrow{p} \theta$ holds. Therefore $\widehat{\theta}_n$ is asymptotically efficient for θ.

2. $X_n = \begin{cases} 0 & p = 1 - \frac{1}{n} \\ n & p = \frac{1}{n} \end{cases}$ and $X \equiv 0$. Then X has cdf $F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$ and $E(X) = 0$.

   (1) Find cdf $F_n(x)$ for X_n.

   Cdf for X_n is $F_n(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{n} & x \in [0,\, n) \\ 1 & x \ge n \end{cases}$.

   (2) Show that $X_n \xrightarrow{d} X$.

   When $x < 0$, $F_n(x) = 0 \longrightarrow 0 = F(x)$.
   When $x > 0$, $F_n(x) = \begin{cases} 1 - \frac{1}{n} & x \in (0,\, n) \\ 1 & x \in [n\, \infty) \end{cases} \longrightarrow 1 = F(x)$.
   Thus $F_n(x) \longrightarrow F(x)$ for all $x \ne 0$. Hence $X_n \xrightarrow{d} X$.

   (3) Show that $E(X_n) \not\to E(X)$.

   $E(X_n) = 0\,\left(1 - \frac{1}{n}\right) + n \times \frac{1}{n} = 1$. But $E(X) = 0$. So $E(X_n) \not\to E(X)$.