Stat871 HW02

1. $\begin{pmatrix} y \\ y_* \end{pmatrix} \sim \left(\begin{pmatrix} X \\ X_* \end{pmatrix} \mu, \begin{pmatrix} \Sigma & C \\ C' & V \end{pmatrix} \right)$.

   (1) Find expression for $\text{Cov}(By - y_*)$.

$$\begin{aligned} \text{Cov}(By - y_*) &= \text{Cov}\left((B, -I) \begin{pmatrix} y \\ y_* \end{pmatrix} \right) = (B, -I) \begin{pmatrix} \Sigma & C \\ C' & V \end{pmatrix} \begin{pmatrix} B' \\ -I \end{pmatrix} \\ &= B\Sigma B' - BC - C'B' + V. \end{aligned}$$

   (2) Suppose $T = B + H(I - XX^+)$, $D = (B\Sigma - C')(I - XX^+)$. Show that

$$\text{Cov}(Ty - y_*) - \text{Cov}(By - y_*) = HD' + DH' + H(I - XX^+)\Sigma(I - XX^+)H'.$$

$$\begin{aligned} & \text{Cov}(Ty - y_*) - \text{Cov}(By - y_*) \\ = & [T\Sigma T' - TC - (TC)' + V] - [B\Sigma B' - BC - (BC)' + V] \\ = & (T - B + B)\Sigma(T - B + B)' - (T - B)C - [(T - B)C]' - B\Sigma B' \\ = & (T - B)\Sigma(T - B)' + (T - B)\Sigma B' + B\Sigma(T - B)' - (T - B)C - [(T - B)C]' \\ = & (T - B)\Sigma(T - B)' + (T - B)(\Sigma B' - C) + (B\Sigma - C')(T - B)' \\ = & (T - B)\Sigma(T - B)' + H[(B\Sigma - C')(I - XX^+)]' + [(B\Sigma - C')(I - XX^+)]H' \\ = & H(I - XX^+)\Sigma(I - XX^+)H' + HD' + DH'. \end{aligned}$$

   (3) Argue that if $(B\Sigma - C')(I - XX^+) = 0$, then $\text{Cov}(Ty - y_*) - \text{Cov}(By - y_*) \geq 0$.

   By (2) with $D = 0$,

$$\text{Cov}(Ty - y_*) - \text{Cov}(By - y_*) = H(I - XX^+)\Sigma(I - XX^+)H'.$$

   But $\Sigma > 0$. So

$$\text{Cov}(Ty - y_*) - \text{Cov}(By - y_*) = [H(I - XX^+)]\Sigma[H(I - XX^+)]' \geq 0.$$

2. $\overline{X}_n$ is the mean of a sample of size n, and μ is the population mean.

   (1) Show that $\frac{n}{n+k}\overline{X}_n$ is an asymptotically unbiased estimator for μ.

   $\lim_{n\to\infty} E\left(\frac{n}{n_k}\overline{X}_n\right) = \lim_{n\to\infty} \frac{n}{n+k}\mu = \mu$.
   So $\frac{n}{n+k}\overline{X}_n$ is an asymptotically unbiased estimator for μ.

   (2) Show that $\frac{n}{n+k}\overline{X}_n$ is a consistent estimator for μ.

$$\begin{cases} \overline{X}_n \xrightarrow{p} \mu \\ \frac{n}{n+k} \longrightarrow 1 \end{cases} \Longrightarrow \begin{pmatrix} \overline{X}_n \\ \frac{n}{n+k} \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \mu \\ 1 \end{pmatrix} \Longrightarrow \frac{n}{n+k}\overline{X}_n \xrightarrow{p} \mu \cdot 1 = \mu.$$

   So $\frac{n}{n+k}\overline{X}_n$ is a consistent estimator for μ.