Stat871 HW01

1. It is well-known that from the distribution $N(\mu,\, \sigma^2)$ with $\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$, the Fisher information $I(\theta) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$; and based on a sample from the distribution $S = \begin{pmatrix} \sum_i x_i \\ \sum_i x_i^2 \end{pmatrix}$ is sufficient and complete for θ.

   (1) Let $\overline{x}$ and s^2 be the sample mean and sample variance. Show that $\widehat{\theta} = \begin{pmatrix} \overline{x} \\ s^2 \end{pmatrix}$ is the best estimator for θ in the class UE(θ) with respect to MSCPE risk.

   $E(\widehat{\theta}) = \begin{pmatrix} E(\overline{x}) \\ E(s^2) \end{pmatrix} = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} = \theta.$

   $\widehat{\theta} = \begin{pmatrix} \overline{x} \\ s^2 \end{pmatrix} = \begin{pmatrix} \frac{\sum x_i}{n} \\ \frac{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}{n-1} \end{pmatrix}$ is a function of sufficient and complete statistics S.

   Thus $\widehat{\theta}$ is the best estimator in UE(θ) by MSCPE risk.

   (2) Show that Cov$(\widehat{\theta}) \neq$ CRLB(θ).
   Hint: $\overline{x} \sim N\left(\mu,\, \frac{\sigma^2}{n}\right)$ and $s^2 \sim \frac{\sigma^2}{n-1}\chi^2(n-1)$ are independent.

   $\text{CRLB}(\theta) = [nI(\theta)]^{-1} = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}.$

   But $\text{Cov}(\widehat{\theta}) = \begin{pmatrix} \text{var}(\overline{x}) & \text{cov}(\overline{x},\, s^2) \\ \text{cov}(s^2,\, \overline{x}) & \text{var}(s^2) \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n-1} \end{pmatrix} \neq \text{CRLB}(\theta).$

2. $y_1, ..., y_n$ is a random sample from a population with mean $\mu \in R^k$ and variance-covariance matrix $\sigma^2 V \in R^{k\times k}$. y_* is the next observation to be taken.
   Using matrices $1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in R^n$, $1_k = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in R^k$, identity $I_n \in R^{n\times n}$, identity $I_k \in R^{k\times k}$ and Kronecker product defined as $A \otimes B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \otimes B \overset{def}{=\!=} \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$ express the followings.

   (1) With $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $E\left[\begin{pmatrix} y \\ y_* \end{pmatrix}\right] = \begin{pmatrix} X \\ X_* \end{pmatrix}\mu$. Find X and X_*.

   $E(y) = \begin{pmatrix} E(y_1) \\ \vdots \\ E(y_n) \end{pmatrix} = \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \begin{pmatrix} I_k \\ \vdots \\ I_k \end{pmatrix}\mu = (1_n \otimes I_k)\mu.$ So $X = 1_n \otimes I_k$

$E(y_*) = \mu = I_k\mu$. So $X_* = I_k$.

(2) $\text{Cov}\left[\begin{pmatrix} y \\ y_* \end{pmatrix}\right] = \sigma^2 \begin{pmatrix} \Sigma & C \\ C' & V \end{pmatrix}$. Find Σ and C.

$$\text{Cov}(y) = \begin{pmatrix} \sigma^2 V & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2 V \end{pmatrix} = \sigma^2 \begin{pmatrix} V & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & V \end{pmatrix} = \sigma^2 (I_n \otimes V). \text{ So } \Sigma = I_n \otimes V.$$

$C = \text{Cov}(y, y_*) = 0 \in R^{nk \times k}$

(3) Let $\overline{y}$ be the sample mean. Then $\overline{y} = By$. Find B.

$$\overline{y} = \frac{y_1 + \cdots + y_n}{n} = \frac{1}{n}(I_k, ..., I_k) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{1}{n}(1_n' \otimes I_k) y. \text{ So } B = \frac{1_n' \otimes I_k}{n}.$$