Stat 871 **Exam 3** **April 16, 2025**

Name: ____

1. Consider constructing uniformly most powerful test on $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$.

   (1) One may use a statistic $T(X)$ where $X = (X_1, ..., X_n)$ is a sample. What is assumption we have to make to find $T(X)$? (10 points)

   The pdf of pmf of sample has the form

   $$f(X,\,\theta) = \exp[p(\theta) + q(X) + \eta(\theta)T(X)]$$

   where $\eta(\theta)$ is a 1-1 function of θ.

   (2) Once $T(X)$ is identified what is the general form of the α-level unbiased UMP test $\phi(T)$? Write the conditions on $\phi(T)$ via expectations. (10 points)

   $$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_i & T = c_i \;\imath = 1,\,2 \\ 0 & c_1 < T < c_2 \end{cases}$$

   where $E_{\theta_0}[\phi(T)] = \alpha$ and $E_{\theta_0}[T\phi(T)] = \alpha E_{\theta_0}(T)$.

   (3) For Geometric population with $f(x;\,\theta) = \theta(1-\theta)^{x-1}$, $0 < \theta < 1$, $x = 1,\,2, ...,$ find $T(X)$. (15 points)

   $$\begin{aligned} f(x_1, .., x_n;\,\theta) &= \theta^n(1-\theta)^{\sum_i x_i - n} = \left(\tfrac{\theta}{1-\theta}\right)^n (1-\theta)^{\sum_i x_i} \\ &= \exp\left[n \ln \tfrac{\theta}{1-\theta} + \ln(1-\theta)\,\textstyle\sum_i x_i\right] \end{aligned}$$

   where $\ln(1-\theta)$ is a 1-1 function of $\theta \in (0,\,1)$. Thus $T(X) = X_1 + \cdots + X_n$.

2. For testing H_0 versus H_a, what do we mean by saying that $\phi(X)$ is locally most powerful test at θ_1 in locally α-level tests at θ_0? (20 points)

   The statement means

   (i) $\theta_0 \in H_0$ and $\phi(X) \in \mathcal{L}_{\theta_0} = \{0 \leq \psi(X) \leq 1 :\, E_{\theta_0}[\phi(X)] \leq \alpha\}$, the collection of all locally α-level tests at θ_0.

   (ii) $\forall\, \psi(X) \in \mathcal{L}_{\theta_0}$, there exists $\epsilon > 0$ such that $H_a \cap (\theta_1 - \epsilon,\, \theta_1 + \epsilon) \neq \emptyset$ and

   $$E_\theta[\psi(X)] \leq E_\theta[\phi(X)] \text{ for all } \theta \in H_a \cap (\theta_1 - \epsilon,\, \theta_1 + \epsilon).$$

3. For $H_0 : \theta = \theta_0$ with one-sided H_a consider locally most powerful α-level test at θ_0 based on sample $X = (X_1, ..., X_n)$.

(1) The test can be expressed as

$$\phi(X) = \begin{cases} 1 & f_2(X) - kf_1(X) > 0 \\ r & f_2(X) - kf_1(X) = 0 \\ 0 & f_2(X) - kf_1(X) < 0 \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha.$$

For $H_a : \theta > \theta_0$ write out $f_1(X)$ and $f_2(X)$.
For $H_a : \theta < \theta_0$ write out $f_1(X)$ and $F_2(X)$. (15 points)

Let $f(X; \theta)$ be the pdf or pmf of sample X.
For $H_a : \theta > \theta_0$, $f_1(X) = f(X; \theta_0)$ and $f_2(X) = f'_\theta(X; \theta_0)$.
For $H_a : \theta < \theta_0$, $f_1(X) = f(X; \theta_0)$ and $f_2(X) = -f'_\theta(X; \theta_0)$.

(2) The two tests can use the same statistic U with different forms of $\phi(U)$.
How was U defined?
For $H_a : \theta > \theta_0$ write out $\phi(U)$.
For $H_a : \theta < \theta_0$ write out $\phi(U)$. (15 points)

With sample $X = (X_1, .., X_n)$, $U = \frac{f'_\theta(X;\theta_0)}{f(X;\theta_0)}$.

$$\text{For } H_a : \theta > \theta_0, \qquad \phi(U) = \begin{cases} 1 & U > c \\ r & U = c \\ 0 & U < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(U)] = \alpha.$$

$$\text{For } H_a : \theta < \theta_0, \qquad \phi(U) = \begin{cases} 1 & U < c \\ r & U = c \\ 0 & U > c \end{cases} \quad \text{with } E_{\theta_0}[\phi(U)] = \alpha.$$

(3) For Geometric population with pmf $f(x; \theta) = \theta(1-\theta)^{x-1}$, $0 < \theta < 1$ and $x = 1, 2, ...$ find U. (15 points)

$f(X; \theta) = \prod_i \theta(1-\theta)^{x_i - 1} = \theta^n (1-\theta)^{\sum x_i - n}$

$f'_\theta(X; \theta) = n\theta^{n-1}(1-\theta)^{\sum x_i - n} + \theta^n(\sum x_i - n)(1-\theta)^{\sum x_i - n - 1}(-1).$

$\frac{f'_\theta(X;\theta)}{f(X;\theta)} = \frac{n}{\theta} - \frac{\sum_i - n}{1-\theta} = \frac{n - \theta \sum x_i}{\theta(1-\theta)}$

Thus $U(X_1, .., X_n) = \frac{n - \theta_0 \sum_{i=1}^n X_i}{\theta_0(1-\theta_0)}$.