Stat871 **Exam 2** **Mar. 26, 2025**

Name:

1. Discrete X has pmf $f(x;\theta)$ where $\theta \in \{\theta_0, \theta_1\}$.

x	1	2	3	4	5
$f(x;\theta_0)$	0.2	0.2	0.2	0.2	0.2
$f(x;\theta_1)$	0.1	0.4	0.1	0.1	0.3

Based on one observation X, for $H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_1$ find most powerful test among all tests with level 0.05. (35 points)

x	1	2	3	4	5
$f(x;\theta_0)$	0.2	0.2	0.2	0.2	0.2
$f(x;\theta_1)$	0.1	0.4	0.1	0.1	0.3
$\Lambda = f_1(x)/f_0(x)$	0.5	2	0.5	0.5	1.5

$$\Phi(X) = \begin{cases} 1 & \Lambda > k \\ r & \Lambda = k \\ 0 & \Lambda < k \end{cases}$$

$0.05 = P_{\theta_0}(\Lambda > K) + r\, P_{\theta_0}(\Lambda = K) = P_{\theta_0}(\Lambda > 2) + r\, P_{\theta_0}(\Lambda = 2) = 0 + r\, P_{\theta_0}(X = 2)$.

So $r = \frac{0.05}{P_{\theta_0}(X=2)} = \frac{0.05}{0.2} = 0.25$.

Thus $\phi(X) = \begin{cases} 0.25 & X = 2 \\ 0 & X = 1, 3, 4, 5 \end{cases}$ is MP test among all tests with level 0.05.

2. Circle TRUE or FALSE, but not both (30 points)

(1) If $E_{\theta_0}[\phi(X)] = \alpha$, then $\phi(X)$ is an α-level test on $H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_1$.

TRUE✓ FALSE

(2) If $E_{\theta_0}[\phi(X)] \le \alpha$, then $\phi(X)$ is an α-level test on $H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_1$.

TRUE✓ FALSE

(3) If $E_{\theta_0}[\phi(X)] = \alpha$, then $\phi(X)$ is an α-level test on $H_0: \theta \le \theta_0$ vs $H_a: \theta > \theta_0$.

TRUE FALSE✓

(4) If $E_{\theta_0}[\phi(X)] = \alpha$, then $\phi(X)$ is an α-level test on $H_0: \theta = \theta_0$ vs $H_a: \theta > \theta_0$.

TRUE✓ FALSE

3. $X_1, ..., X_n$ is a random sample from $N(0, \sigma^2)$.

(1) Find the likelihood function and identify statistic T such that the likelihood function has monotone likelihood ratio in T. (13 points)

The likelihood function is

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}(\sigma^2)^{1/2}} \exp\left(-\frac{1}{2\sigma^2}X_i^2\right) = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_i X_i^2\right).$$

With $\sigma_1^2 < \sigma_2^2$, the likelihood ratio

$$\Lambda = \frac{L(\sigma_2^2)}{L(\sigma_1^2)} = \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{n/2} \exp\left[\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\sum_i X_i^2\right]$$

is an increasing function of $T = \sum_i X_i^2$. So $L(\sigma^2)$ has MLR in $T = \sum_i X_i^2$.

(2) Find the distribution for T you found in (1). (10 points)

$X_i \sim N(0, \sigma^2) \Longrightarrow \frac{X_i}{\sigma} \sim N(0, 1^2) \Longrightarrow \frac{X_i^2}{\sigma^2} \sim \chi^2(1) \Longrightarrow \sum_{i=1}^n \frac{X_i^2}{\sigma^2} \sim \chi^2(n)$.

Thus $\frac{T}{\sigma^2} \sim \chi^2(n)$. Hence $T = \sum_{i=1}^n X_i^2 \sim \sigma^2\chi^2(n)$.

(3) For $H_0 : \sigma^2 \leq 4$ vs $H_a : \sigma^2 > 4$ find α-level UMP test. (12 points)

Let $\phi(X) = \begin{cases} 1 & T = \sum_{i=1}^n X_i^2 > k \\ 0 & T = \sum_{i=1}^n X_i^2 < k \end{cases}$

$$\alpha = E_{\sigma^2=4}[\phi(X)] = P_{\sigma^2=4}(T > k) = P(4\chi^2(n) > k) = P\left(\chi^2(n) > \frac{k}{4}\right),$$

So $\frac{k}{4} = \chi^2_\alpha(n) \Longrightarrow k = 4\chi^2_\alpha(n)$.

Thus $\phi(X) = \begin{cases} 1 & \sum_{i=1}^n X_i^2 > 4\chi^2_\alpha(n) \\ 0 & \sum_{i=1}^n X_i^2 < 4\chi^2_\alpha(n) \end{cases}$ is α-level UMP test.