Stat871 **Exam 1** **Feb. 17, 2025**

Name:

1. $X \sim \text{Bernoulli}(p)$.

   (1) Find Fisher information $I(p)$ from the distribution of X. (10 points)

   $f(x;\, p) = p^x(1-p)^{1-x}$, $x = 0,\, 1.$ $\ln f(x;\, p) = x\ln p + (1-x)\ln(1-p)$

   $\frac{d}{dp}\ln f(X;\, p) = \frac{X}{p} - \frac{1-X}{1-p} = \frac{X-p}{p(1-p)} = \frac{X}{p(1-p)} - \frac{1}{1-p} \sim \left(0,\, \frac{1}{p(1-p)}\right)$.

   Thus $I(p) = \frac{1}{p(1-p)}$.

   (2) Let $X_1, ..., X_n$ be a random sample from X. By CLT find a statistic with asymptotically normal distribution. (5 points)

   By CLT, $\overline{X}_n = \frac{X_1+\cdots+X_n}{n} \sim AN\left(p,\, \frac{p(1-p)}{n}\right)$.

   (3) Based on the result in (2) find an asymptotically normal estimator for $\theta = \text{var}(X)$. (20 points)

   $\theta = \text{var}(X) = p(1-p) = \theta(p)$ is a function of p. $\frac{d\theta}{dp} = 1 - 2p$

   With $\overline{X}_n \sim AN\left(p, \frac{p(1-p)}{n}\right)$ and $g(x) = x(1-x)$, by Delta method

   $$\overline{X}_n\left(1 - \overline{X}_n\right) \sim AN\left(p(1-p),\, (1-2p)^2\frac{p(1-p)}{n}\right), \text{ i.e.,}$$

   $\overline{X}_n - \overline{X}_n^2 \sim AN\left(\theta,\, \frac{(1-2p)^2p(1-p)}{n}\right)$ is an asymptotically normal estimator for $\theta = \text{var}(X) = p(1-p)$.

   (4) Show that the estimator in (3) for θ is best asymptotically normal estimator.(15 points)

   $\text{CRLB}(\theta) = \text{CRLB}(p(1-p)) = \left[\frac{d}{dp}p(1-p)\right]^2[nI(p)]^{-1} = (1-2p)^2\frac{p(1-p)}{n}$. So

   $$\overline{X}_n(1 - \overline{X}_n) \sim AN(\theta,\, \text{CRLB}(\theta)).$$

   Hence $\overline{X}_n(1-\overline{X}_n)$ is a best asymptotically normal estimator for θ.

2. $X_1, ..., X_n$ is a random sample from $X \sim U(0,\, \theta)$. Find asymptotically normal distribution for $2\overline{X}_n$. (10 points)

$X \sim U(0,\, \theta) \Longrightarrow X \sim \left(\frac{\theta}{2},\, \frac{\theta^2}{12}\right)$.

By CLT, $\overline{X}_n \sim AN\left(\frac{\theta}{2},\, \frac{\theta^2}{12n}\right)$.

So, $2\overline{X}_n \sim AN\left(2\frac{\theta}{2},\, \frac{2^2\theta^2}{12n}\right) = AN\left(\theta,\, \frac{\theta^2}{3n}\right)$.

3. Let A1: $E(\widehat{\theta}_n) = \theta$ A2: $\text{Cov}(\widehat{\theta}_n) = \text{CRLB}(\theta)$.
B1: $\widehat{\theta}_n \xrightarrow{p} \theta$ B2: $\text{cov}[\sqrt{n}\widehat{\theta}_n] \longrightarrow I^{-1}(\theta)$.
Check TRUE or FALSE for the following statements.

(1) A1+A2$\Longrightarrow$ $\widehat{\theta}_n$ is asymptotically efficient estimator for θ. (10 points)

TRUE✓ FALSE

(2) B1+B2$\Longrightarrow$ $\widehat{\theta}_n$ is asymptotically efficient estimator for θ. (10 points)

TRUE✓ FALSE

(3) A1+B2$\Longrightarrow$ $\widehat{\theta}_n$ is asymptotically efficient estimator for θ. (10 points)

TRUE✓ FALSE

(4) B1+A2$\Longrightarrow$ $\widehat{\theta}_n$ is asymptotically efficient estimator for θ. (10 points)

TRUE✓ FALSE