Stat871 HW01

1. It is well-known that from the distribution $N(\mu, \sigma^2)$ with $\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$, the Fisher information $I(\theta) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$; and based on a sample from the distribution $S = \begin{pmatrix} \sum_i x_i \\ \sum_i x_i^2 \end{pmatrix}$ is sufficient and complete for θ.

   (1) Let $\overline{x}$ and s^2 be the sample mean and sample variance. Show that $\widehat{\theta} = \begin{pmatrix} \overline{x} \\ s^2 \end{pmatrix}$ is the best estimator for θ in the class UE(θ) with respect to MSCPE risk.

   (2) Show that $\text{Cov}(\widehat{\theta}) \neq \text{CRLB}(\theta)$.
   Hint: $\overline{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $s^2 \sim \frac{\sigma^2}{n-1} \chi^2(n-1)$ are independent.

2. $y_1, ..., y_n$ is a random sample from a population with mean $\mu \in R^k$ and variance-covariance matrix $\sigma^2 V \in R^{k \times k}$. y_* is the next observation to be taken.
   Using matrices $1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in R^n$, $1_k = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in R^k$, identity $I_n \in R^{n \times n}$, identity $I_k \in R^{k \times k}$ and Kronecker product defined as $A \otimes B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \otimes B \overset{def}{=\!=} \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$ express the followings.

   (1) With $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $E\left[\begin{pmatrix} y \\ y_* \end{pmatrix}\right] = \begin{pmatrix} X \\ X_* \end{pmatrix} \mu$. Find X and X_*.

   (2) $\text{Cov}\left[\begin{pmatrix} y \\ y_* \end{pmatrix}\right] = \sigma^2 \begin{pmatrix} \Sigma & C \\ C' & V \end{pmatrix}$. Find Σ and C.

   (3) Let $\overline{y}$ be the sample mean. Then $\overline{y} = By$. Find B.