L08 Convergence with probability 1

1. Preliminaries

   (1) In mathematics

   $$\begin{aligned} \text{as } n \longrightarrow \infty,\ X_n \longrightarrow X &\iff X_n - X \longrightarrow 0 \iff \|X_n - X\| \longrightarrow 0 \\ &\iff \forall\, \epsilon > 0\ \exists\, N > 0\ \forall\, n > N\ \ \|X_n - X\| < \epsilon \\ &\iff \forall\, k \geq 1\ \exists\, N \geq 1\ \forall\, n \geq N\ \ \|X_n - X\| < \tfrac{1}{k}. \end{aligned}$$

   In above expressions $< \epsilon$ and $< \frac{1}{k}$ can be replaced by $\leq \epsilon$ and $\leq \frac{1}{k}$.
   $X_n \longrightarrow X \iff (X_n) \longrightarrow g(X)$ for all continuous $g(\cdot)$.

   (2) In basic probability theory

   $$\begin{aligned} &P(\cap_t A_t) = 1 \iff P(A_t) = 1\,\forall\, t \\ &P(\cup_t A_t) = 0 \iff P(A_t) = 0\,\forall t \\ &A_1 \subset A_2 \subset \cdots \Longrightarrow \cup_t A_t \overset{def}{=\!=} \lim A_t \Longrightarrow P(\lim A_t) = \lim P(A_t) \\ &A_1 \supset A_2 \supset \cdots \Longrightarrow \cap_t A_t \overset{def}{=\!=} \lim A_t \Longrightarrow P(\lim A_t) = \lim P(A_t) \end{aligned}$$

   (3) Random variables and vectors
   Random X is a mapping from Ω to R or R^k.
   For $B \subset R$ or R^k, let $D = \{w \in \Omega : X(w) \in B\} \subset \Omega$. Then $P(X \in B) = P(D)$.

2. Convergence with probability 1

   (1) A set in Ω on which $X_n \longrightarrow X$

   $$\begin{aligned} &w \in \cap_{k=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} [w \in \Omega :\ \|X_n(w) - X(w)\| < \tfrac{1}{k}] \\ &\iff \forall k \geq 1\ w \in \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} [w \in \Omega :\ \|X_n(w) - X(w)\| < \tfrac{1}{k}] \\ &\iff \forall k \geq 1\ \exists N \geq 1\ w \in \cap_{n=N}^{\infty} [w \in \Omega :\ \|X_n(w) - X(w)\| < \tfrac{1}{k}] \\ &\iff \forall k \geq 1\ \exists N \geq 1\ \forall n \geq N\ \|X_n(w) - X(w)\| < \tfrac{1}{k} \\ &\iff X_n(w) \longrightarrow X(w). \end{aligned}$$

   Thus $\cap_{k=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} [w \in \Omega :\ \|X_n(w) - X(w)\| < \frac{1}{k}] = [w \in \Omega :\ X_n(w) \longrightarrow X(w)]$.
   In above expressions "$< \frac{1}{k}$" can be replaced by "$\leq \frac{1}{k}$".

   (2) Definition

   $$\begin{aligned} & X_n \xrightarrow{wp1} X \overset{def}{\iff} X_n \xrightarrow{a.s.} X \overset{def}{\iff} P(X_n \longrightarrow X) = 1 \\ \iff\ & P\left[\cap_{k=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} \left(w \in \Omega :\ \|X_n(w) - X(w)\| < \tfrac{1}{k}\right)\right] = 1 \\ \iff\ & P\left[\cup_{N=1}^{\infty} \cap_{n=N}^{\infty} \left(w \in \Omega :\ \|X_n(w) - X(w)\| < \tfrac{1}{k}\right)\right] = 1\ \forall\, k \geq 1. \end{aligned}$$

   In above expressions "$< \frac{1}{k}$" can be replaced by "$\leq \frac{1}{k}$".

   (3) A sufficient and necessary condition

   $$X_n \xrightarrow{wp1} X \iff \lim_{N \to \infty} P\left[\cap_{n=N}^{\infty} \left(w \in \Omega :\ \|X_n(w) - X(w)\| < \frac{1}{k}\right)\right] = 1\ \forall\, k \geq 1.$$

Proof: Let $T_N = \cap_{n=N}^{\infty}[w \in \Omega : \|X_n(w) - X(w)\| < \frac{1}{k}]$.

By (2) $X_n \xrightarrow{wp1} X \Longleftrightarrow P(\cup_{N=1}^{\infty} T_N) = 1\,\forall k$.

But $T_1 \subset T_2 \subset \cdots$. Thus $P(\cup_{N=1}^{\infty} T_N) = P(\lim_N T_N) = \lim_N P(T_N)$. Therefore

$$\begin{aligned} X_n \xrightarrow{wp1} X &\Longleftrightarrow P(U_{N=1}^{\infty} T_N) = 1\,\forall k \geq 1 \Longleftrightarrow \lim_N P(T_N) = 1\,\forall\, k \geq 1 \\ &\Longleftrightarrow \lim_{N\to\infty} P\left[\cap_{n=N}^{\infty}\left(w \in \Omega : \|X_n(w) - X(w)\| < \tfrac{1}{k}\right)\right] = 1\ \forall\, k. \end{aligned}$$

Ex1: Note that $P(A) = 1 \Longleftrightarrow P(A^c) = 0$. Thus

$$\begin{aligned} & X_n \xrightarrow{wp1} X \Longleftrightarrow X_n \xrightarrow{a.s.} X \Longleftrightarrow P(A) = 1 \Longleftrightarrow P(A^c) = 0 \\ \Longleftrightarrow\ & P\left[\cup_{k=1}^{\infty} \cap_{N=1}^{\infty} \cup_{n=N}^{\infty}\left(w \in \Omega : \|X_n(w) - X(w)\| \geq \tfrac{1}{k}\right)\right] = 0 \\ \Longleftrightarrow\ & P\left[\cap_{N=1}^{\infty} \cup_{n=N}^{\infty}\left(w \in \Omega : \|X_n(w) - X(w)\| \geq \tfrac{1}{k}\right)\right] = 0\ \forall\, k \geq 1 \\ \Longleftrightarrow\ & \lim_{N\to\infty} P\left[\cup_{n=N}^{\infty}\left(w \in \Omega : \|X_n(w) - X(w)\| \geq \tfrac{1}{k}\right)\right] = 0\ \forall\, k \geq 1. \end{aligned}$$

In above expressions "$\geq \frac{1}{k}$" can be replaced by "$> \frac{1}{k}$".

Ex2: $X_n \xrightarrow{wp1} X \Longleftrightarrow X_n - X \xrightarrow{wp1} 0 \Longleftrightarrow \|X_n - X\| \xrightarrow{wp1} 0$.

3. Sufficient condition and properties

(1) Borel-Cantelli Lemma

If for all $\epsilon > 0$ $\sum_{n=1}^{\infty} P\left(\|X_n - X\| > \epsilon\right) < \infty$, then $X_n \xrightarrow{wp1} X$.

Proof:

$$\begin{aligned} & \textstyle\sum_{n=1}^{\infty} P(\|X_n - X\| > \epsilon) < \infty\ \forall \epsilon > 0 \\ \Longleftrightarrow\ & \textstyle\sum_{n=1}^{\infty} P\left(\|X_n - X\| \geq \frac{1}{k}\right) < \infty\ \forall k \geq 1 \\ \Longleftrightarrow\ & \lim_{N\to\infty} \textstyle\sum_{n=N}^{\infty} P\left(\|X_n - X\| \geq \frac{1}{k}\right) = 0\ \forall k \geq 1 \\ \Longrightarrow\ & \lim_{N\to\infty} P\left(\cup_{n=N}^{\infty} \|X_n - X\| \geq \frac{1}{k}\right) = 0\ \forall k \geq 1 \\ \Longleftrightarrow\ & \lim_{N\to\infty} P\left(\cap_{n=N}^{\infty} \|X_n - X\| < \frac{1}{k}\right) = 1\ \forall k \geq 1 \Longleftrightarrow X_n \xleftrightarrow{wp1} X. \end{aligned}$$

Ex3: If X_n has a subsequence X_{n_k} such that $P\left(\|X_{n_k} - X\| > \frac{1}{k}\right) \leq \frac{1}{2^k}$, then $\sum_{k=1}^{\infty} P\left(\|X_{n_k} - X\| > \frac{1}{k}\right) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$. So $X_{n_k} \xrightarrow{as} X$ as $k \to \infty$

(2) A property

$X_n \xrightarrow{a.s} X \Longleftrightarrow g(X_n) \xrightarrow{a.s.} g(X)$ for all continuous $g(\cdot)$.

Ex4: $X_n \xrightarrow{as} X$ implies $AX_n + b \xrightarrow{as} AX + b$.

Ex5: $X_n = \begin{pmatrix} X_{1n} \\ \vdots \\ X_{pn} \end{pmatrix} \xrightarrow{as} X = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$. Then $X_{in} \xrightarrow{as} X_i$ as $n \to \infty$.