L25: Factor models

1. Factor model for $\mathbf{x}$

   (1) Model
   Regression relates vector of interest $\mathbf{y} \in R^p$ to $\mathbf{x} \in R^q$ by $\mathbf{y} = B'\mathbf{x} + \epsilon$.
   When $\mathbf{x} \in R^p$ is of interest, without other vectors to be to related to, let $\mathbf{F} \sim (0, I_q)$ and write

   $$\mathbf{x} = \mu + LF + \epsilon,\ \epsilon \sim (0,\ \Psi),\ \Psi = \text{diag}(\psi_1, .., \psi_p) \text{ and } \text{Cov}(F,\ \epsilon) = 0.$$

   (2) Parameters
   For $\mathbf{x} \sim (\mu,\ \Sigma)$, by factor model $\mathbf{x} \sim (\mu,\ LL' + \Psi)$, i.e., $\Sigma = LL' + \Psi$.
   With $x_i = \mu_i + l_{i1}F_1 + \cdots + l_{iq}F_q + \epsilon_i$, $E(x_i) = \mu_i$ and $\text{var}(x_i) = l_{i1}^2 + \cdots + l_{iq}^2 + \psi_i$.

i	x_i	$\text{var}(x_i)$	F_1	$\cdots$	F_q	h_i^2	ψ_i
1	x_1	σ_1^2	l_{11}^2	$\cdots$	l_{1q}^2	h_1^2	ψ_1
2	x_2	σ_2^2	l_{21}^2	$\cdots$	l_{2q}^2	h_2^2	ψ_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	x_p	σ_p^2	l_{p1}^2	$\cdots$	l_{pq}^2	h_p^2	ψ_p
Sum		$\text{tr}(\Sigma)$	f_1^2	$\cdots$	f_q^2	$\text{tr}(LL')$	$\text{tr}(\Psi)$

   l_{ij}^2 is the contribution of F_j to $\text{var}(x_i)$ via l_{ij} in L;
   $f_j^2 = \sum_i l_{ij}^2$ is contribution of F_j to the total variance in $\mathbf{x}$, $\text{tr}(\Sigma)$;
   The ith communality $h_i^2 = \sum_j l_{ij}^2$ is the contribution of F to $\text{var}(x_i)$.
   $\text{tr}(LL') = \text{tr}(L'L) = \sum_i h_i^2 = \sum_j f_j^2$ is the contribution of F to the total variance of $\mathbf{x}$.
   ψ_i is the part of $\text{var}(x_i)$ unexplained by F.
   $\text{tr}(\Psi) = \sum_i \psi_i$ is the part of total variance in $\mathbf{x}$ unexplained by F.

   Ex1: From

i	x_i	σ_i^2	F_1	F_2	h_i^2	ψ_i
1	x_1	19	16	1	17	2
2	x_2	57	49	4	53	4
3	x_3	38	1	36	37	1
4	x_4	68	1	64	65	3
Sum		182	67	105	172	10

   The proportion of total variance in $\mathbf{x}$ explained by F is $\frac{\text{tr}(LL')}{\text{tr}(\Sigma)} = \frac{172}{182} = 94.51\%$.
   The contribution from F to $\text{var}(x_3)$ is $h_3^2 = 37$ and the contribution from F_2 to the total variance in x is $f_2^2 = 105$.

2. Factor model for $\mathbf{z}$, standardized x

   (1) Model
   For $\mathbf{x} \sim (\mu,\ \Sigma)$, $\mathbf{z} = V^{-1/2}(\mathbf{x} - \mu) \sim (0,\ \rho)$ is standardized $\mathbf{x}$.

   $$\mathbf{z} = L_zF + \epsilon_z \text{ is a factor model if}$$

   $F \sim (0,\ I_q)$, $\epsilon_z \sim (0,\ \Psi_z)$, $\Psi_z = \text{diag}(\psi_{z1}, .., \psi_{zp})$ and $\text{Cov}(F,\ \epsilon_z) = 0$.

   (2) Parameters
   For $\mathbf{z} \sim (0,\ \rho)$ by $\mathbf{z} = L_zF + \epsilon_z$, $\rho = L_zL_z' + \Psi_z$
   With $z_i = l_{zi1}F_1 + \cdots + l_{ziq}F_q + \epsilon_{zi}$, $E(z_i) = 0$ and $1 = \text{var}(z_i) = l_{zi1}^2 + \cdots + l_{ziq}^2 + \psi_{zi}$.

i	z_i	$\text{var}(z_i)$	F_1	$\cdots$	F_q	h_{zi}^2	ψ_{zi}
1	z_1	1	l_{z11}^2	$\cdots$	l_{z1q}^2	h_{z1}^2	ψ_{z1}
2	z_2	1	l_{z21}^2	$\cdots$	l_{z2q}^2	h_{z2}^2	ψ_{z2}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	z_p	1	l_{zp1}^2	$\cdots$	l_{zpq}^2	h_{zp}^2	ψ_{zp}
Sum		p	f_{*1}^2	$\cdots$	f_{*q}^2	$\text{tr}(L_zL_z')$	$\text{tr}(\Psi_z)$

Ex2: Factor model is not unique. For example with $\mathbf{z} = L_z F + \epsilon_z$ and orthogonal $Q \in R^{q\times q}$, let $L_* = L_z Q$ and $F_* = Q'F$. Then $\mathbf{z} = L_z F + \epsilon_z = L_* F_* + \epsilon_z$ where $F_* \sim (0, Q'Q) = (0, I_q)$ and $\mathrm{Cov}(F_*, \epsilon_z) = Q'0 = 0$.

3. Converting models

(1) Converting a factor model for $\mathbf{x}$ to that for $\mathbf{z}$.
Suppose $\mathbf{x} = \mu + LF + \epsilon$ has parameter table in (2) of 1.
Then $\mathbf{z} = V^{-1/2}(\mathbf{x} - \mu) = V^{-1/2}(LF + \epsilon) = (V^{-1/2}L)F + (V^{-1/2}\epsilon) = L_z F + \epsilon_z$.
Here $L_z = V^{-1/2}L = \left(l_{ij}/\sqrt{\mathrm{var}(x_i)}\right)_{p\times q}$, $l_{zij}^2 = l_{ij}^2/\mathrm{var}(x_i)$, $h_{zi}^2 = h_i^2/\mathrm{var}(x_i)$.
$\epsilon_z = V^{-1/2}\epsilon \sim (0, V^{-1/2}\Psi V^{-1/2}) = (0, V^{-1}\Psi)$, $\psi_{zi} = \psi_i/\mathrm{var}(x_i)$. So

i	z_i	ρ_{ii}	F_1	$\cdots$	F_q	h_{zi}^2	ψ_{zi}
1	z_1	1	l_{11}^2/σ_1^2	$\cdots$	l_{1q}^2/σ_1^2	h_1^2/σ_1^2	ψ_1/σ_1^2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	z_p	1	l_{p1}^2/σ_p^2	$\cdots$	l_{pq}^2/σ_p^2	h_p^2/σ_p^2	ψ_p/σ_p^2
Sum		p	f_{*1}^2	$\cdots$	f_{*q}^2	$\mathrm{tr}(L_z L_z')$	$\mathrm{tr}(\Psi_z)$

For this conversion $\sigma_i^2 = \mathrm{var}(x_i)$, $i = 1, .., p$, are required.

(2) Converting a factor model for $\mathbf{z}$ to that for $\mathbf{x}$
Suppose $\mathbf{z} = L_z F + \epsilon_z$ has parameter table in (3) of 1.
Then $\mathbf{x} = V^{1/2}\mathbf{z} + \mu = \mu + V^{1/2}(L_z F + \epsilon_z) = \mu + (V^{1/2}L_z)F + (V^{1/2}\epsilon_z) = \mu + LF + \epsilon$.
Here $L = V^{1/2}L_z = \left(l_{zij}\cdot\sqrt{\mathrm{var}(x_i)}\right)_{p\times q}$, $l_{ij}^2 = l_{zij}^2 \cdot \mathrm{var}(x_i)$, $h_i^2 = h_{zi}^2 \cdot \mathrm{var}(x_i)$.
$\epsilon = V^{1/2}\epsilon_z \sim (0, V^{1/2}\Psi_z V^{1/2}) = (0, V\Psi_z)$, $\psi_i = \psi_{zi}\cdot \mathrm{var}(x_i)$. So

i	x_i	σ_i^2	F_1	$\cdots$	F_q	h_{zi}^2	ψ_{zi}
1	x_1	$1\cdot\sigma_1^2$	$l_{z11}^2\cdot\sigma_1^2$	$\cdots$	$l_{z1q}^2\cdot\sigma_1^2$	$h_{z1}^2\cdot\sigma_1^2$	$\psi_{z1}\cdot\sigma_1^2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	x_p	$1\cdot\sigma_p^2$	$l_{zp1}^2\cdot\sigma_p^2$	$\cdots$	$l_{zpq}^2\cdot\sigma_p^2$	$h_{zp}^2\cdot\sigma_p^2$	$\psi_{zp}\cdot\sigma_p^2$
Sum		$\mathrm{tr}(\Sigma)$	f_1^2	$\cdots$	f_q^2	$\mathrm{tr}(LL')$	$\mathrm{tr}(\Psi)$

Note that $l_{ij} = \sqrt{\mathrm{var}(x_i)}l_{zij}$, $l_{ij}^2 = \mathrm{var}x_i l_{zij}^2$, $h_i^2 = \mathrm{var}(x_i)h_{zi}^2$ $\psi_i = \mathrm{var}(x_i)\psi_{zi}$.

(3) Orthogonal factor model
A factor model is orthogonal if the columns of its loading matrix are orthogonal.
If $\mathbf{x} = \mu + LF + \epsilon$ is orthogonal, then $L'L = \mathrm{diag}(f_1^2, ..., f_q^2)$ and the converted factor model for $mathbfz$ may not be orthogonal.
If $\mathbf{z} = L_z F + \epsilon_z$ is orthogonal, then $L_z' L_z = \mathrm{diag}(f_{*1}^2, ..., f_{*q}^2)$ and the converted factor model for $\mathbf{x}$ may not be orthogonal.

L26 Estimating parameters in factor model

1. Estimating parameters in factor model for $\mathbf{z}$

   (1) A rationale

   For $\mathbf{x} \sim (\mu, \Sigma)$, $\mathbf{z} \sim (0, \rho)$. By EVD $\rho = P_* \Lambda_* P_*' = (P_{*I}, P_{*II}) \begin{pmatrix} \Lambda_{*I} & 0 \\ 0 & \Lambda_{*II} \end{pmatrix} P_*'$ where P_{*I} has q columns. Then $\mathbf{y}_* = P_*'\mathbf{z}$ is a vector of principal components of $\mathbf{z}$.
   So $\mathbf{z} = P_*\mathbf{y}_* = P_{*I}\mathbf{y}_{*I} + P_{*II}\mathbf{y}_{*II}$ where y_{*I} contains first q principal components of $\mathbf{z}$.
   Let $P_{*I}\mathbf{y}_{*I} = (P_{*I}\Lambda_{*I}^{1/2})(\Lambda_{*I}^{-1/2}\mathbf{y}_{*I}) = L_zF$. Then $F = \Lambda_{*I}^{-1/2}y_{*I} \sim (0, I_q)$ and
   $\mathrm{tr}[\mathrm{Cov}(L_zF)] = \mathrm{tr}[\mathrm{tr}(P_{*I}\mathbf{y}_{*I})] = \mathrm{tr}(\Lambda_{*I})$ is the majority of $p = \mathrm{tr}(\rho) = \mathrm{tr}(\Lambda_*)$.
   By Factor model $\mathbf{z} = L_zF + \epsilon_z$, modify ρ to $L_zL_z' + \Psi_z$ where $\Psi_z = \mathrm{diag}(\psi_{z1}, .., \psi_{zp})$ with $\psi_{zi} = 1 - \sum_j l_{zij}^2$.

   (2) Estimated parameters

   With a sample from $\mathbf{x}$, sample correlation matrix $\mathbf{R}$ is taken as initial ρ. Following (1) one can get estimated parameters for the model $\mathbf{z} = L_zF + \epsilon_z$.
   Because L_z is estimated by $P_{*I}\Lambda_{*I}^{1/2}$ with orthogonal columns, the model is orthogonal one and hence $f_{*j}^2 = \lambda_{*j}$, $j = 1, .., q$.

   (3) SAS

```
proc factor nfactor=2;
        var x1 x2 x3;
        run;
```

   produces four tables via EVD of $\mathbf{R}$.

   (i) Eigenvalue table

EV	Diff	Propor	Cumula
λ_1	$\lambda_1 - \lambda_2$	$\lambda_1/3$	$\lambda_1/3$
λ_2	$\lambda_2 - \lambda_3$	$\lambda_2/3$	$(\lambda_1 + \lambda_2)/3$
λ_3		$\lambda_3/3$	1

   Comments: $\lambda_1 + \lambda_2 + \lambda_3 = 3 = \mathrm{tr}(\rho)$; $\lambda_1 + \lambda_2 = \sum_i h_{zi}^2 = \sum_j f_{*j}^2$; $\lambda_1 = f_{*1}^2$ and $\lambda_2 = f_{*2}^2$.

   (ii) Loading matrix table

	Factor 1	Factor 2
X1	l_{z11}	l_{z12}
X2	l_{z21}	l_{z22}
X3	l_{z31}	l_{z32}

   Comments: $L_z = P_{*I}\Lambda_{*I}^{1/2}$ has orthogonal columns. With known Λ_{*I}, P_{*I} can be calculated.

   (iii) f_{*j}^2 table

Factor 1	Factor 2
$f_{*1}^2 = \lambda_1$	$f_{*2}^2 = \lambda_2$

   (iv) Communality table

X1	X2	X3
h_{z1}^2	h_{z2}^2	h_{z3}^2

   Comment: Based on this output $\psi_{zi} = 1 - h_{zi}^2$, $i = 1, ..., p$.

   Without $\mathrm{var}(x_i)$ from this output one can not bet parameters for the factor model for $\mathbf{x}$.

2. Estimating parameters in factor model for $\mathbf{x}$

   (1) A rationale

   For $\mathbf{x} \sim (\mu, \Sigma)$, by EVD $\Sigma = P\Lambda P' = (P_I, P_{II}) \begin{pmatrix} \Lambda_I & 0 \\ 0 & \Lambda_{II} \end{pmatrix} P'$ where P_I has q columns. Then $\mathbf{y} = P'\mathbf{x}$ is a vector of principal components of $\mathbf{x}$.
   So $\mathbf{x} = P\mathbf{y} = P_I\mathbf{y}_I + P_{II}\mathbf{y}_{II}$ where y_I contains first q principal components of $\mathbf{x}$.
   Let $P_I\mathbf{y}_I = (P_I\Lambda_I^{1/2})(\Lambda_I^{-1/2}\mathbf{y}_I) = LF$. Then $F = \Lambda_I^{-1/2}y_I \sim (0, I_q)$ and
   $\mathrm{tr}[\mathrm{Cov}(LF)] = \mathrm{tr}[\mathrm{tr}(P_I\mathbf{y}_I)] = \mathrm{tr}(\Lambda_I)$ is the majority of $\sum_i \mathrm{var}(x_i) = \mathrm{tr}(\Sigma) = \mathrm{tr}(\Lambda)$.
   By Factor model $\mathbf{x} = LF + \epsilon$, modify Σ to $LL' + \Psi$ where $\Psi = \mathrm{diag}(\psi_1, .., \psi_p)$ with $\psi_i = \mathrm{var}(x_i) - \sum_j l_{ij}^2$.

(2) Estimated parameters

With a sample from $\mathbf{x}$, sample S_u is taken as initial ρ. Following (1) one can get estimated parameters for the model $\mathbf{x} = LF + \epsilon$.

Because L is estimated by $P_I \Lambda_I^{1/2}$ with orthogonal columns, the model is orthogonal one and hence $f_j^2 = \lambda_j$, $j = 1, .., q$.

(3) SAS: An example

```
proc factor nfactor=2 cov;
     var x1 x2 x3;
     run;
```

```
Eigenvalues of the Covariance Matrix: Total = 24.262  Average = 8.0873333
         Eigenvalue    Difference    Proportion    Cumulative
   1    23.3034906    22.7048000        0.9605        0.9605
   2     0.5986906     0.2388719        0.0247        0.9852
   3     0.3598188                      0.0148        1.0000

                     Factor Pattern
                Factor1         Factor2
       x1         0.99102        -0.03682
       x2         0.97229        -0.18174
       x3         0.96927         0.23427

            Variance Explained by Each Factor
          Factor        Weighted    Unweighted
         Factor1     23.3034906    2.86695344
         Factor2      0.5986906    0.08926935

   Total Communality: Weighted = 23.902181   Unweighted = 2.956223
          Variable     Communality       Weight
           x1           0.98348129    11.0720000
           x2           0.97837588     6.4170000
           x3           0.99436561     6.7730000
```

The output has four tables.

(i) In Eigenvalue table

$\lambda_1 + \lambda_2 + \lambda_3 = \sum_i \text{var}(x_i)$; $\lambda_1 + \lambda_2 = \sum_{i=1}^3 h_i^2 = \sum_{j=1}^2 f_j^2$; $\lambda_1 = f_1^2$ and $\lambda_2 = f_2^2$.

From this table Λ, Λ_I can be obtained.

(ii) In Factor Pattern table

Loading matrix L_z is given. Note that $L_z = V^{-1/2} L$ and $L = P_I \Lambda_I^{1/2}$.

So with V given in table (iv) and Λ_I given in table (i), L and P_I can be calculated.

(iii) In table for the Variance Explained by Each Factor

under "Weighted" f_j^2, $j = 1, .., q$, are listed. Under "Unweighted" f_{*j}^2, $j = 1, .., q$, are listed.

$\mathbf{x} = LF + \epsilon$ is an orthogonal model. So $f_j^2 = \lambda_j$, $j = 1, ..., q$.

(iv) In table 4 under "Communality" are the estimated h_{zj}^2 for the model for Z.

Under "Weight" are the estimated values for $\text{var}(x_i)$, $i = 1, .., p$.