

L13: One-sample t -tests

1. α -level t -tests

- (1) Elements of a test scheme

Null hypothesis H_0 and alternative hypothesis H_a ; Test statistic; Decision rule; Computation; Conclusion.

- (2) Decision rule and significance level α

Type I error: Rejecting H_0 , but H_0 is true; Type II error: Accepting H_0 , but H_0 is false.

If $P(\text{Type I error}) = P(\text{Rejection of } H_0 | H_0 \text{ is true}) \leq \alpha$, then α is called the significance level.

α -level test has decision rule such that the significance level is α .

- (3) t -tests

Suppose $\frac{\hat{\theta} - \theta}{s_{\hat{\theta}}} \sim t(k)$, then

$$H_0 : \theta = \theta_0 \text{ vs } H_a : \theta \neq \theta_0$$

$$\text{Test statistic: } t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$$

$$\text{Reject } H_0 \text{ if } t < -t_{\alpha/2}(k) \text{ or } t > t_{\alpha/2}(k)$$

$$H_0 : \theta \leq \theta_0 \text{ vs } H_a : \theta > \theta_0$$

$$\text{Test statistic: } t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$$

$$\text{Reject } H_0 \text{ if } t > t_{\alpha}(k)$$

$$H_0 : \theta \geq \theta_0 \text{ vs } H_a : \theta < \theta_0$$

$$\text{Test statistic: } t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$$

$$\text{Reject } H_0 \text{ if } t < -t_{\alpha}(k)$$

are α -level test.

Proof Consider the test with upper-sided H_a .

$$\begin{aligned} P(\text{Type I error}) &= P(t > t_{\alpha}(k) | H_0) = P\left(\frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}} > t_{\alpha}(k) | \theta \leq \theta_0\right) \\ &\leq P\left(\frac{\hat{\theta} - \theta}{s_{\hat{\theta}}} > t_{\alpha}(k)\right) = P(t(k) > t_{\alpha}(k)) = \alpha. \end{aligned}$$

- (4) One-sample t -tests

Based on one sample from $N(\mu, \Sigma)$, let $\theta = l'\mu$, $\hat{\theta} = l'\bar{x}$ and $s_{\hat{\theta}} = s_{l'\bar{x}} = \sqrt{\frac{l'S_0 l}{n}}$.

Then $\frac{\hat{\theta} - \theta}{s_{\hat{\theta}}} \sim t(n-1)$. So one can have three α -level tests on θ .

Ex1: A sample of size 25 from $N(\mu, \Sigma)$ has

$$\bar{x} = \begin{pmatrix} 185.72 \\ 151.12 \\ 183.84 \\ 149.24 \end{pmatrix} \text{ and symmetric } S = \begin{pmatrix} 91.482 & 50.754 & 66.875 & 44.267 \\ & 52.186 & 49.259 & 33.651 \\ & & 96.774 & 54.278 \\ & & & 43.222 \end{pmatrix}$$

Are μ_1 and μ_3 equal? Support your conclusion by a test at the level 0.05.

$$H_0 : \mu_1 - \mu_3 = 0 \text{ versus } H_a : \mu_1 - \mu_3 \neq 0$$

$$\text{Test statistic: } t = \frac{\bar{x}_1 - \bar{x}_3}{s_{\bar{x}_1 - \bar{x}_3}}$$

$$\text{Reject } H_0 \text{ if } t < -2.0639 \text{ or } t > 2.0639 \text{ for } \alpha = 0.05$$

$$\bar{x}_1 - \bar{x}_3 = 1.88, s_{\bar{x}_1 - \bar{x}_3} = 1.507; t = 1.24$$

Fail to reject H_0 .

Data did not show evidence against $\mu_1 = \mu_3$.

2. Tests by p -value

- (1) Observed significance level

p -value is a probability calculated from the observed value of the test statistic such that

$$\alpha\text{-level test rejects } H_0 \iff p\text{-value} < \alpha.$$

p -value is also called observed significance level. When using computer software, p -value but not rejection regions will be produced.

- (2) Test by p -value

Test by p -value has five steps: Write out H_0 and H_a ; Define test statistic; Give the formula for p -value; Computation; Conclusion.

$H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$ Test statistic: $t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$ p-value: $2 \times P(t(k) > t_{ob})$

$H_0 : \theta \leq \theta_0$ vs $H_a : \theta > \theta_0$ Test statistic: $t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$ p-value: $P(t(k) > t_{ob})$	$H_0 : \theta \geq \theta_0$ vs $H_a : \theta < \theta_0$ Test statistic: $t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}}$ p-value: $P(t(k) < t_{ob})$
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Proof Consider the test with lower-sided H_a .

$$P(t(k) < t_{ob}) < \alpha \iff P(t(k) < t_{ob}) < P(t(k) < -t_{\alpha}(k)) \iff t_{ob} < -t_{\alpha}(k) \\ \iff \alpha\text{-level test on } H_0 : \theta \geq \theta_0 \text{ vs } H_a : \theta < \theta_0 \text{ rejects } H_0.$$

So $P(t(k) < t_{ob})$ is the observed significance level.

Ex2: Redo Ex1 using p -value

$H_0 : \mu_1 - \mu_3 = 0$ versus $H_a : \mu_1 - \mu_3 \neq 0$ Test statistic: $t = \frac{\bar{x}_1 - \bar{x}_3}{s_{\bar{x}_1 - \bar{x}_3}}$ p-value: $2 \times P(t(n-1) > t_{ob})$. $t = 1.24$, p-value: $2 \times P(t(24) > 1.24) = 0.2270$ Fail to reject H_0 . Data did not show evidence against $\mu_1 = \mu_3$.

Ex3: State your conclusion on the test in Ex2 if $\alpha = 0.05, 0.10, 0.20, 0.30$.

With p -value = 0.2270, for $\alpha = 0.05, 0.10, 0.20$, we fail to reject H_0 ;

But for $\alpha = 0.30$, we reject H_0 .

3. Relation to CIs

- (1) θ_0 is in $1 - \alpha$ CI for $\theta \iff \alpha$ -level test with two-sided H_a fails to reject H_0
- (2) θ_0 is in $1 - \alpha$ upper-sided CI for $\theta \iff \alpha$ -level test with upper-sided H_a fails to reject H_0
- (3) θ_0 is in $1 - \alpha$ lower-sided CI for $\theta \iff \alpha$ -level test with lower-sided H_a fails to reject H_0

Proof θ_0 is in $1 - \alpha$ lower-sided CI for $\theta \iff -\infty < \theta_0 < \hat{\theta} + t_{\alpha}(k)s_{\hat{\theta}}$
 $\iff t = \frac{\hat{\theta} - \theta_0}{s_{\hat{\theta}}} > -t_{\alpha}(k) \iff \alpha$ -level test on $H_0 : \theta \geq \theta_0$ vs $H_a : \theta < \theta_0$ fails to reject H_0 .

Ex4: In Ex3 is 0 in a 95% 90%, 80% and 70% confidence interval for θ ?

0 is in a 95%, 90%, 80% confidence interval for θ since a test on $H_0 : \theta = 0$ vs $H_a : \theta \neq 0$ fails to reject H_0 at $\alpha = 0.05, 0.10, 0.20$.

0 is not in a 70% confidence interval for θ since a test on $H_0 : \theta = 0$ vs $H_a : \theta \neq 0$ rejects H_0 at $\alpha = 0.20$.