Stat776 **HW01**

1. 1.4.2 without (i)-(iv) p22
   Let $\mathbf{S}(\alpha) = \frac{1}{n}\sum_{r=1}^{n}(\mathbf{x}_r - \alpha)(\mathbf{x}_r - \alpha)'$. Show that $\mathbf{S}(\alpha) = \mathbf{S} + (\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)'$.
   Hint: Write $\mathbf{x}_r - \alpha = \mathbf{x}_r - \overline{\mathbf{x}} + \overline{\mathbf{x}} - \alpha$.

$$\begin{aligned}
\mathbf{S}(\alpha) &= \tfrac{1}{n}\textstyle\sum_{r=1}^{n}(\mathbf{x}_r - \alpha)(\mathbf{x}_r - \alpha)' = \tfrac{1}{n}\sum_{r=1}^{n}(\mathbf{x}_r - \overline{\mathbf{x}} + \overline{\mathbf{x}} - \alpha)(\mathbf{x}_r - \overline{\mathbf{x}} + \overline{\mathbf{x}} - \alpha)' \\
&= \tfrac{1}{n}\textstyle\sum_{r=1}^{n}(\mathbf{x}_r - \overline{\mathbf{x}})(\mathbf{x}_r - \overline{\mathbf{x}})' + \tfrac{1}{n}\sum_{r=1}^{n}(\mathbf{x}_r - \overline{\mathbf{x}})(\overline{\mathbf{x}} - \alpha)' \\
&\qquad + \tfrac{1}{n}\textstyle\sum_{r=1}^{n}(\overline{\mathbf{x}} - \alpha)(\mathbf{x}_r - \overline{\mathbf{x}})' + \tfrac{1}{n}\sum_{r=1}^{n}(\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)' \\
&= \mathbf{S} + \mathbf{0}(\overline{\mathbf{x}} - \alpha)' + (\overline{\mathbf{x}} - \alpha)\mathbf{0}' + (\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)' \\
&= \mathbf{S} + (\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)'
\end{aligned}$$

2. 1.4.2 (ii) p22 Show that $\mathrm{tr}(\mathbf{S}) = \min_\alpha \mathrm{tr}[\mathbf{S}(\alpha)]$.
   Hint: $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.

   By the result in 1.4.2 and formulas $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(CD) = \mathrm{tr}(DC)$

$$\begin{aligned}
\mathrm{tr}[\mathbf{S}(\alpha)] &= \mathrm{tr}[\mathbf{S} + (\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)'] = \mathrm{tr}(\mathbf{S}) + \mathrm{tr}[(\overline{\mathbf{x}} - \alpha)(\overline{\mathbf{x}} - \alpha)'] \\
&= \mathrm{tr}(\mathbf{S}) + \mathrm{tr}[(\overline{\mathbf{x}} - \alpha)'(\overline{\mathbf{x}} - \alpha)] = \mathrm{tr}(\mathbf{S}) + \|\overline{\mathbf{x}} - \alpha\|^2 \geq \mathrm{tr}(\mathbf{S}).
\end{aligned}$$

   Thus $\mathrm{tr}(\mathbf{S}) = \min_\alpha \mathrm{tr}[\mathbf{S}(\alpha)]$.

3. 1.5.1 p23

   Data matrix $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_n' \end{pmatrix} = (\mathbf{x}_{(1)}, .., \mathbf{x}_{(p)}) \in R^{n\times p}$. Let $\mathbf{M} = (m_{ij})_{p\times p} = \mathbf{X}'\mathbf{X} \in R^{p\times p}$.

   (1) Show that $m_{ij} = \mathbf{x}_{(i)}'\mathbf{x}_{(j)}$ and $m_{ii} = \mathbf{x}_{(i)}'\mathbf{x}_{(i)}$.

$$\mathbf{M} = \mathbf{X}'\mathbf{X} = \begin{pmatrix} \mathbf{x}_{(1)}' \\ \vdots \\ \mathbf{x}_{(p)}' \end{pmatrix} (\mathbf{x}_{(1)}, .., \mathbf{x}_{(p)}).$$ So $m_{ij} = \mathbf{x}_{(i)}'\mathbf{x}_{(j)}$. With $i = j$, $m_{ii} = \mathbf{x}_{(i)}'\mathbf{x}_{(i)}$.

   (2) Show that $m_{ij} = n(s_{ij} + \overline{x}_i\,\overline{x}_j)$ and $m_{ii} = n(s_{ii} + \overline{x}_i^2)$.
   Hint: Express $n\mathbf{S}$ via $\mathbf{M}$ and $\overline{x}$ and take elements (i, j).

   $\mathbf{S} = \mathbf{X}'\frac{\mathbf{H}}{n}\mathbf{X}$. So $n\mathbf{S} = \mathbf{X}'\mathbf{H}\mathbf{X}$. But $\mathbf{H} = \mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n'}{n}$.

   So $n\mathbf{S} = \mathbf{X}'\mathbf{X} - \frac{\mathbf{X}'\mathbf{1}_n\mathbf{1}_n'\mathbf{X}}{n} = \mathbf{M} - n\overline{\mathbf{x}}\,\overline{\mathbf{x}}'$.

   Take (i, j) elements of both sides. $ns_{ij} = m_{ij} + n\overline{x}_i\,\overline{x}_j$.

   Therefore $m_{ij} = n(s_{ij} - \overline{x}_i\,\overline{x}_j)$. With $i = j$, $m_{ii} = n(s_{ii} - \overline{x}_i^2$.