Quiz 1

1. Z_t is a sequence of independent standard normal random variables.
Let $X_t = \begin{cases} Z_t & t \text{ is odd} \\ aZ_{t-1}^2 + b & t \text{ is even} \end{cases}$

(1) Find the mean function of X_t. (5 points)

$$E(X_t) = \begin{cases} E(Z_t) & = & 0 & t \text{ is odd} \\ E(aZ_{t-1}^2 + b) & = & a + b & t \text{ is even} \end{cases}$$

(2) Find the autocovariance function of X_t. (5 points)

$$\gamma(0) = \text{var}(X_t) = \begin{cases} \text{var}(Z_t) & = & 1 & t \text{ is odd} \\ \text{var}(aZ_{t-1}^2 + b) & = & 2a^2 & t \text{ is even} \end{cases}$$

$$\gamma(1) = \begin{cases} \text{cov}(X_{2t}, X_{2t+1}) & = & \text{cov}(aZ_{2t-1}^2 + b, Z_{2t+1}) & = & 0 \\ \text{cov}(X_{2t-1}, X_{2t}) & = & \text{cov}(Z_{2t-1}, aZ_{2t-1}^2 + b) & = & 0 \end{cases}$$

$$\gamma(-1) = \begin{cases} \text{cov}(X_{2t}, X_{2t-1}) & = & \text{cov}(aZ_{2t-1}^2 + b, Z_{2t-1}) & = & 0 \\ \text{cov}(X_{2t+1}, X_{2t}) & = & \text{cov}(Z_{2t+1}, aZ_{2t-1}^2 + b) & = & 0 \end{cases}$$

$\gamma(h) = 0$ when $|h| > 1$.

(3) Is X_t a strictly stationary time series? (5 points)

No. If X_t is strictly stationary, then $X_1 = Z_1 \sim N(0, 1^2)$ and $Z_2 = aZ_1^2 + b \sim a\chi^2(1) + b$ would have had the same distribution. But they do not.

(4) Is X_{2t+1} a strictly stationary time series? (5 points)

Yes. $X_{2t+1} = Z_{2t+1}$ are iid $N(0, 1^2)$ and hence is strictly stationary.

(5) Is X_{2t} a stationary time series? (5 points)

$X_{2t} = aZ_{2t-1}^2 + b$ are iid $a\chi^2(1) + b$ and hence is strictly stationary.
Also $\text{var}(X_{2t}) = 2a^2 < \infty$. So $E(X_{2t}^2) < \infty$. Thus X_{2t} is stationary.

(6) Find conditions on a and b such that X_t is a white noise. (5 points)

If $a = \frac{1}{\sqrt{2}}$ and $b = -\frac{1}{\sqrt{2}}$, then X_t represents an uncorrelated series with mean 0 and variance 1, and hence is a white noise.

(7) Under the conditions in (6) are X_t and X_{t+1} independent? (5 points)

No. $X_2 = \frac{1}{\sqrt{2}}Z_1^2 - \frac{1}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\chi^2(1) - \frac{1}{\sqrt{2}}$.
But $X_2|(X_1 = 1) = 0 \not\sim \frac{1}{\sqrt{2}}\chi^2(1) - \frac{1}{\sqrt{2}}$. So X_1 and X_2 are not independent.

(8) Under the conditions in (6) are X_t and X_{t+1} identically distributed? (5 points)

No. $X_1 = Z_1 \sim N(0, 1)$, but $X_2 = \frac{1}{\sqrt{2}}Z_1^2 - \frac{1}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\chi^2(1) - \frac{1}{\sqrt{2}}$.

2. X_t is stationary with mean μ_X and ACVF $\gamma_X(h)$. Y_t is stationary with mean μ_Y and ACVF $\gamma_Y(h)$. X_t and Y_t are independent, i.e., X_r and Y_t are independent.

(1) Is $X_t + Y_t$ stationary? If yes, find its mean and ACVF. (10 points)

Yes $X_t + Y_t$ is stationary with $E(X_t + Y_t) = \mu_X + \mu_Y$ and
$\gamma_{X+Y}(h) = \text{cov}(X_t + Y_t,\, X_{t+h} + Y_{t+h}) = \gamma_X(h) + \gamma_Y(h)$.

(2) Is $X_t - Y_t$ stationary? If yes, find its mean and ACVF. (10 points)

Yet $X_t - Y_t$ is stationary with $E(X_t - Y_t) = \mu_X - \mu_Y$ and
$\gamma_{X-Y}(h) = \text{cov}(X_t - Y_t,\, X_{t+h} - Y_{t+h}) = \gamma_X(h) + \gamma_Y(h)$.

(3) Is $X_t \cdot Y_t$ stationary? If yes, find its mean and ACVF. (10 points)

Yes $X_t \cdot Y_t$ is stationary with $E(X_t \cdot Y_t) = \mu_X \cdot \mu_Y$ and

$$\begin{aligned}
\gamma_{XY}(h) &= \text{cov}(X_tY_t,\, X_{t+h}Y_{t+h}) = E(X_tY_tX_{t+h}Y_{t+h}) - E(X_tY_t)E(X_{t+h}Y_{t+h}) \\
&= E(X_tX_{t+h})E(Y_tY_{t+h}) - \mu_X\mu_Y\mu_X\mu_Y \\
&= [\gamma_X(h) + \mu_X^2][\gamma_Y(h) + \mu_Y^2] - \mu_X^2\mu_Y^2 \\
&= \mu_X^2\gamma_Y(h) + \mu_Y^2\gamma_X(h) + \gamma_X(h)\gamma_Y(h).
\end{aligned}$$

3. X_1, X_2, X_4, X_5 are observed from $X_t = \phi X_{t-1} + Z_t \sim \text{AR}(1)$
where $Z_t \sim \text{WN}0,\, \sigma^2)$ and $0 < |\phi| < 1$.

(1) Find the best linear predictor P_2X_8 of X_8 using X_1 and X_2.
What is the corresponding mean squared error? (15 points)

$X_t = \phi X_{t-1} + Z_t \sim \text{AR}(1) \Longrightarrow E(X_t) = 0$ and $\gamma(h) = \frac{\phi^h\sigma^2}{1-\phi^2}$.

$\mathbf{\Gamma}_2 = \text{Cov}\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right) = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}$ and $\gamma_2 = \text{Cov}\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, X_8\right) = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} \phi^7 \\ \phi^6 \end{pmatrix}$

$\mathbf{a}_2 = \mathbf{\Gamma}_2^{-1}\gamma_2 = \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}^{-1}\begin{pmatrix} \phi^7 \\ \phi^6 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi^6 \end{pmatrix}$. So $P_2X_8 = \mathbf{a}_2'\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \phi^6X_2$.

The mean squared error: $\gamma(0) - \mathbf{a}_2'\gamma_2 = \frac{\sigma^2}{1-\phi^2} - (0,\, \phi^6)\frac{\sigma^2}{1-\phi^2}\begin{pmatrix} \phi^7 \\ \phi^6 \end{pmatrix} = \frac{1-\phi^{12}}{1-\phi^2}\,\sigma^2$.

(2) Find the best linear estimate of the missing value X_3 using X_4 and X_5.
Compute the mean squared error of this estimate. (15 points)

$X_t = \phi X_{t-1} + Z_t \sim \text{AR}(1) \Longrightarrow E(X_t) = 0$ and $\gamma(h) = \frac{\phi^h\sigma^2}{1-\phi^2}$.

$\mathbf{\Gamma}_2 = \text{Cov}\left(\begin{pmatrix} X_4 \\ X_5 \end{pmatrix}\right) = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}$ and $\gamma_2 = \text{Cov}\left(\begin{pmatrix} X_4 \\ X_5 \end{pmatrix}, X_3\right) = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} \phi \\ \phi^2 \end{pmatrix}$

$\mathbf{a}_2 = \mathbf{\Gamma}_2^{-1}\gamma_2 = \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}^{-1}\begin{pmatrix} \phi \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$. So $P_2X_3 = \mathbf{a}_2'\begin{pmatrix} X_4 \\ X_5 \end{pmatrix} = \phi X_4$.

The mean squared error: $\gamma(0) - \mathbf{a}_2'\gamma_2 = \frac{\sigma^2}{1-\phi^2} - (\phi,\, 0)\frac{\sigma^2}{1-\phi^2}\begin{pmatrix} \phi \\ \phi^2 \end{pmatrix} = \sigma^2$.