Stat775 HW10

1. 4.5 p117
Stationary X_t has ACVF $\gamma_X(h)$ and spectral distribution function $F_X(\lambda)$.
Stationary Y_t has ACVF $\gamma_Y(h)$ and spectral distribution function $F_Y(\lambda)$.
X_t and Y_t are uncorrelated. Show that $Z_t = X_t + Y_t$ is stationary with ACVF $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$ and spectral distribution function $F_Z(\lambda) = F_X(\lambda) + F_Y(\lambda)$.

   First $E(Z_t) = E(X_t) + E(Y_t)$ does not depend on t.
   Because in $Z_t = X_t + Y_t$, X_t and Y_t are uncorrelated,

$$\begin{aligned} \text{cov}(Z_{t+h}, Z_t) &= \text{cov}(X_{t+h} + Y_{t+h}, X_t + Y_t) = \text{cov}(X_{t+h}, X_t) + \text{cov}(Y_{t+h}, Y_t) \\ &= \gamma_X(h) + \gamma_Y(h). \end{aligned}$$

   So Z_t is stationary with $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$. From

$$\begin{aligned} \gamma_Z(h) &= \gamma_X(h) + \gamma_Y(h) = \int_{-\pi}^{\pi} e^{i\lambda h} dF_X(\lambda) + \int_{-\pi}^{\pi} e^{i\lambda h}\, dF_Y(\lambda) \\ &= \int_{-\pi}^{\pi} e^{i\lambda h}\, d\left[F_X(\lambda) + F_Y(\lambda)\right] \end{aligned}$$

   $F_Z(\lambda) = F_X(\lambda) + F_Y(\lambda)$ is the spectral distribution function for Z_t.

2. 4.9 p118
Let $D_+ = \left(\frac{\pi}{6} - 0.01, \frac{\pi}{6} + 0.01\right)$, $D_- = \left(\frac{-\pi}{6} - 0.01, \frac{-\pi}{6} + 0.01\right)$ and $D = D_- \cup D_+$.
X_t has spectral density $f_X(\lambda) = \begin{cases} 100 & \lambda \in D \\ 0 & \text{otherwise} \end{cases}$.

   (a) Find ACVF of X_t at $h = 0, 1$.

   $\gamma_X(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_X(\lambda)\, d\lambda = 200 \int_{D_+} \cos(h\lambda)\, d\lambda$.
   When $h = 0$, $\gamma_X(0) = 200 \int_{D_+} d\lambda = 4$. When $h = 1$,

$$\begin{aligned} \gamma_X(1) &= 200 \int_{D_+} \cos(\lambda)\, d\lambda = 200 \left[\sin\left(\tfrac{\pi}{6} + 0.01\right) - \sin\left(\tfrac{\pi}{6} - 0.01\right)\right] \\ &= 200 \times 2\cos(\pi/6)\sin(0.01) = 200\sqrt{3}\sin(0.01). \end{aligned}$$

   (b) Find spectral density for $Y_t = \Delta_{12} X_t = X_t - X_{t-12}$.

   The power transfer function for $Y_t = (I - B^{12})X_t$ is

$$\beta(\lambda) = (1 - e^{-i12\lambda})(1 - e^{i12\lambda}) = 2 - 2\cos(12\lambda).$$

   So the spectral density for Y_t is

$$f_Y(\lambda) = \beta(\lambda) f_X(\lambda) = \begin{cases} 200\left[1 - \cos(12\lambda)\right] & \lambda \in D \\ 0 & \text{otherwise} \end{cases}.$$

(c) Find the variance of Y_t.

$$\begin{aligned} \gamma_Y(0) &= \int_{-\pi}^{\pi} 1 \cdot f_Y(\lambda)\, d\lambda = 400 \int_{D_+} [1-\cos(12\lambda)]\, d\lambda \\ &= 8 - \tfrac{200}{3}\sin(0.12). \end{aligned}$$

(d) Sketch $\beta(\lambda)$ in (b) and explain the effects of the filter near 0 and near $\frac{\pi}{6}$.

$\beta(\lambda) = 2 - 2\cos(12\lambda)$ gives a periodic curve with values between 0 and 2, and period $\frac{\pi}{6}$. $\beta(0) = \beta(\pi/6) = 0$. So near 0 and $\pi/6$ the filter reduces the spectral density of Y_t to the values close to 0.

3. 4.10 p118
$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$, $Z_t \sim \mathrm{WN}(0, \sigma^2)$ where $|\phi| > 1$ and $|\theta| > 1$.

Define $\widetilde{\phi}(B) = 1 - \frac{1}{\phi}B$, $\widetilde{\theta}(B) = 1 + \frac{1}{\theta}B$ and $W_t = \widetilde{\theta}^{-1}(B)\,\widetilde{\phi}(B)X_t$.

(a) Show that W_t has a constant spectral density function.

$\phi(B) = I - \phi B$ has power transfer function

$$|\phi(e^{-i\lambda\cdot})|^2 = (1-\phi e^{-i\lambda})(1-\phi e^{i\lambda}) = 1 + \phi^2 - 2\phi\cos(\lambda).$$

$\theta(B) = 1 + \theta B$ has power transfer function

$$|\theta(e^{-i\lambda\cdot})|^2 = (1+\theta e^{-i\lambda})(1+\theta e^{i\theta}) = 1 + \theta^2 + 2\theta\cos(\lambda).$$

$\widetilde{\phi}(B) = 1 - \frac{1}{\phi}B$ has power transfer function

$$|\widetilde{\phi}(e^{-i\lambda\cdot})|^2 = 1 + \frac{1}{\phi^2} - \frac{2}{\phi}\cos(\lambda) = \frac{1+\phi^2-2\phi\cos(\lambda)}{\phi^2}.$$

$\widetilde{\theta}(B) = 1 + \frac{1}{\theta}B$ has power function

$$|\widetilde{\theta}(e^{-i\lambda\cdot})|^2 = 1 + \frac{1}{\theta^2} + \frac{2}{\theta}\cos(\lambda) = \frac{1+\theta^2+2\theta\cos(\lambda)}{\theta^2}.$$

$\widetilde{\theta}^{-1}(B)$ has power transfer function

$$\frac{1}{|\widetilde{\theta}(e^{-i\lambda\cdot})|^2} = \frac{\theta^2}{1+\theta^2+2\theta\cos(\lambda)}.$$

From $\phi(B)X_t = \theta(B)Z_t$, $\quad f_X(\lambda) = \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} f_Z(\lambda) = \frac{1+\theta^2+2\theta\cos(\lambda)}{1+\phi^2-2\phi\cos(\lambda)}\,\frac{\sigma^2}{2\pi}$.
In $W_t = \widetilde{\theta}^{-1}(B)\widetilde{\phi}(B)X_t$, the power transfer function $\beta(\lambda)$ for $\widetilde{\theta}^{-1}(B)\widetilde{\phi}(B)$ is

$$\beta(\lambda) = |\widetilde{\theta}^{-1}(e^{-i\lambda\cdot})|^2 |\widehat{\phi}(e^{-i\lambda\cdot})|^2 = \frac{\theta^2\left(1+\phi^2-2\phi\cos(\lambda)\right)}{\phi^2\left(1+\theta^2+2\theta\cos(\lambda)\right)}.$$

Thus $f_W(\lambda) = \beta(\lambda) f_X(\lambda)$ is the constant $\frac{\theta^2\sigma^2}{2\pi\phi^2}$.

(b) Show that $W_t \sim \mathrm{WN}(0,\, \sigma_W^2)$ and find the expression for σ_W^2.

Note that $E(W_t) = 0$ and

$$\gamma_W(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_W(\lambda)\, d\lambda = \frac{\theta^2\sigma^2}{\phi^2} \int_{-\pi}^{\pi} e^{i\lambda h}\, d\lambda = \begin{cases} \frac{\theta^2\sigma^2}{\phi^2} & h = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Thus $W_t \sim \mathrm{WN}(0,\, \sigma_W^2)$ with $\sigma_W^2 = \frac{\theta^2\sigma^2}{\phi^2}$.

(c) Show that $\widetilde{\phi}(B)X_t = \widetilde{\theta}(B)W_t$ and the model is causal and invertible.

$\widetilde{\theta}^{-1}(B)\widetilde{\phi}(B)X_t = W_t$ implies $\widetilde{\theta}(B)\widetilde{\theta}^{-1}(B)\widetilde{\phi}(B)X_t = \widetilde{\theta}(B)W_t$, i.e.,

$$\widetilde{\phi}(B)X_t = \widetilde{\theta}(B)W_t.$$

$\widetilde{\phi}(z) = 0 \Longrightarrow 1 - \frac{z}{\phi} = 0 \Longrightarrow |z| = |\phi| > 1$. So the model is causal.
$\widetilde{\theta}(z) = 0 \Longrightarrow 1 + \frac{z}{\theta} = 0 \Longrightarrow |z| = |\theta| > 1$. So the model is invertible.