Stat775 **HW09**

1. 4.4 p117

   Is $\gamma(h) = \begin{cases} 1 & h = 0 \\ -0.5 & h = \pm 2 \\ -0.25 & h = \pm 3 \\ 0 & \text{otherwise} \end{cases}$ an ACVF for a stationary X_t?

   Note that (i) $\gamma(-h) = \gamma(h)$ and (ii) $\sum_h |\gamma(h)| = 1 + 2 \times (0.5 + 0.25) = 2.5 < \infty$.
   Let $f(\lambda) = \frac{1}{2\pi}\sum_h e^{-ih\lambda}\gamma(h)$. Then

   $$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\left[e^0 \cdot 1 + (e^{-i2\lambda} + e^{i2\lambda})(-0.5) + (e^{-i3\lambda} + e^{i3\lambda})(-0.25)\right] \\ &= \frac{1}{2\pi}\left[1 - \cos(2\lambda) - \frac{1}{2}\cos(3\lambda)\right]. \end{aligned}$$

   But $f(0) = \frac{1}{2\pi}\left(1 - 1 - \frac{1}{2}\right) = -\frac{1}{4\pi} < 0$.
   Thus $\gamma(h)$ is not an ACVF for a stationary X_t.

2. 4.6 p117

   $Z_t \sim \mathrm{WN}(0, \sigma^2)$, $Y_t = Z_t + 2.5Z_{t-1}$ and $X_t = A\cos\left(\frac{\pi t}{3}\right) + B\sin\left(\frac{\pi t}{3}\right) + Y_t$ where $A \sim (0, v^2)$, $B \sim (0, v^2)$ and Z_t are uncorrelated.

   (1) Find ACVF for X_t.

   Let $S_t = A\cos\left(\frac{\pi t}{3}\right) + B\sin\left(\frac{\pi t}{3}\right)$. Then $X_t = S_t + Y_t$.

   $$\begin{aligned} \gamma_X(h) &= \mathrm{cov}(X_{t+h}, X_t) = \mathrm{cov}(S_{t+h} + Y_{t+h}, S_t + Y_t) = \mathrm{cov}(S_{t+h}, S_t) + \mathrm{cov}(Y_{t+h}, Y_t) \\ &= \gamma_S(h) + \gamma_Y(h). \end{aligned}$$

   $$\begin{aligned} \text{But } \gamma_S(h) &= \mathrm{cov}\left(S_{t+h}, S_t\right) \\ &= \mathrm{cov}\left(A\cos\left(\frac{\pi(t+h)}{3}\right) + B\sin\left(\frac{\pi(t+h)}{3}\right), A\cos\left(\frac{\pi t}{3}\right) + B\sin\left(\frac{\pi t}{3}\right)\right) \\ &= v^2\cos\left(\frac{\pi h}{3}\right) \end{aligned}$$

   and $\gamma_Y(h) = \mathrm{cov}(Y_{t+h}, Y_t) = \mathrm{cov}(Z_{t+h} + 2.5Z_{t+h-1}, Z_t + 2.5Z_{t-1})$

   $$= \begin{cases} (1 + 2.5^2)\sigma^2 & h = 0 \\ 2.5\sigma^2 & h = \pm 1 \\ 0 & |h| > 1 \end{cases}.$$

   So the ACVF for X_t is $\gamma_X(h) = \gamma_S(h) + \gamma_Y(h) = \begin{cases} v^2 + 7.25\sigma^2 & h = 0 \\ 0.5v^2 + 2.5\sigma^2 & h = \pm 1 \\ v^2\cos\left(\frac{\pi h}{3}\right) & |h| > 1 \end{cases}$

   (2) Find spectral distribution function for X_t.

   Let $F_S(\lambda)$ and $F_Y(\lambda)$ be the spectral distribution functions for S_t and Y_t. Then

   $$\begin{aligned} \gamma_X(h) &= \gamma_S(h) + \gamma_Y(h) = \int_{-\pi}^{\pi} e^{i\lambda h} dF_S(\lambda) + \int_{-\pi}^{\pi} e^{i\lambda h} dF_Y(\lambda) \\ &= \int_{-\pi}^{\pi} e^{i\lambda h} d(F_S(\lambda) + F_Y(\lambda)). \end{aligned}$$

So $F_X(\lambda) = F_S(\lambda) + F_Y(\lambda)$.
But $\gamma_S(h) = v^2 \cos\left(\frac{\pi h}{3}\right) = \left(e^{-i\frac{\pi h}{3}} + e^{i\frac{\pi h}{3}}\right) \frac{v^2}{2} = \int_{-\pi}^{\pi} e^{i\lambda h} \, dF_S(\lambda)$

$$\text{where } F_S(\lambda) = \begin{cases} 0 & \lambda < \frac{-\pi}{3} \\ \frac{v^2}{2} & -\frac{\pi}{3} \leq \lambda < \frac{\pi}{3} \\ v^2 & \frac{\pi}{3} \leq \lambda \end{cases} \quad \text{is the spectral distribution function for } S_t.$$

The spectral density for Y_t is $f_Y(\lambda) = \frac{1}{2\pi} \sum_h e^{-i\lambda h} \gamma_Y(h) = \frac{1}{2\pi} \left[7.25\sigma^2 + 5\sigma^2 \cos(\lambda)\right]$.
So the spectral distribution function for Y_t is

$$F_Y(\lambda) = \int_{-\pi}^{\lambda} f_Y(\lambda) \, d\lambda = \frac{\sigma^2}{2\pi} \left[7.25(\lambda + \pi) + 5\sin(\lambda)\right].$$

The spectral distribution function for X_t is $F_X(\lambda) = F_S(\lambda) + F_Y(\lambda)$.