Stat774 HW08

1. 3.6 p95

Show that MA(1) $X_t = Z_t + \theta Z_{t-1}$ with $Z_t \sim \text{WN}(0,\, \sigma^2)$ and MA(1) $Y_t = \widetilde{Z}_t + \frac{1}{\theta}\widetilde{Z}_{t-1}$ with $\widetilde{Z}_t \sim \text{WN}(0,\, \sigma^2\theta^2)$ where $0 < |\theta| < 1$ have the same ACVF.

$$\begin{aligned}
\gamma_X(h) &= \text{cov}(X_{t+|h|},\, X_t) = \text{cov}(Z_{t+|h|} + \theta Z_{t+|h|-1},\, Z_t + \theta Z_{t-1}) \\
&= \begin{cases} \text{cov}(Z_t + \theta Z_{t-1},\, Z_t + \theta Z_{t-1}) = \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2 & h = 0 \\ \text{cov}(Z_{t+1} + \theta Z_t,\, Z_t + \theta Z_{t-1}) = \theta\sigma^2 & |h| = 1 \\ \text{cov}(Z_{t+|h|} + \theta Z_{t+|h|-1},\, Z_t + \theta Z_{t-1}) = 0 & |h| > 1 \end{cases} \\
\gamma_Y h) &= \text{cov}(Y_{t+|h|},\, Y_t) = \text{cov}\left(\widetilde{Z}_{t+|h|} + \tfrac{1}{\theta}\widetilde{Z}_{t+|h|-1},\, \widetilde{Z}_t + \tfrac{1}{\theta}\widetilde{Z}_{t-1}\right) \\
&= \begin{cases} \text{cov}\left(\widetilde{Z}_t + \frac{1}{\theta}\widetilde{Z}_{t-1},\, \widetilde{Z}_t + \frac{1}{\theta}\widetilde{Z}_{t-1}\right) = \sigma^2\theta^2 + \frac{1}{\theta^2}\sigma^2\theta^2 = (1+\theta^2)\sigma^2 & h = 0 \\ \text{cov}\left(\widetilde{Z}_{t+1} + \frac{1}{\theta}\widetilde{Z}_t,\, \widetilde{Z}_t + \frac{1}{\theta}\widetilde{Z}_{t-1}\right) = \frac{1}{\theta}\sigma^2\theta^2 = \theta\sigma^2 & |h| = 1 \\ \text{cov}\left(\widetilde{Z}_{t+|h|} + \frac{1}{\theta}\widetilde{Z}_{t+|h|-1},\, \widetilde{Z}_t + \frac{1}{\theta}\widetilde{Z}_{t-1}\right) = 0 & |h| > 1 \end{cases}
\end{aligned}$$

Clearly $\gamma_Z(h) \equiv \gamma_Y(h)$.

2. 3.7 p95

With $|\theta| > 1$, non-invertible $X_t = Z_t + \theta Z_{t-1}$ where $Z_t \sim \text{WN}(0,\, \sigma^2)$ and $W_t = \sum_{j=0}^{\infty}(-\theta)^{-j}X_{t-j}$.

(i) Show that $W_t \sim \text{WN}(0,\, \sigma_W^2)$ and find the expression of σ_W^2.

$$X_t = Z_t + \theta Z_{t-1} \text{ with } Z_t \sim \text{WN}(0,\, \sigma^2) \Longrightarrow \gamma_X(h) = \begin{cases} (1+\theta^2)\sigma^2 & h = 0 \\ \theta\sigma^2 & |h| = 1 \\ 0 & |h| > 1 \end{cases}$$

$$\begin{aligned}
\gamma_W(0) &= \text{cov}(W_t,\, W_t) \\
&= \text{cov}\big((-\theta)^0 X_t + (-\theta)^{-1}X_{t-1} + (-\theta)^{-2}X_{t-2} + \cdots, \\
&\qquad\qquad (-\theta)^0 X_t + (-\theta)^{-1}X_{t-1} + (-\theta)^{-2}X_{t-2} + \cdots\big) \\
&= \left[(-\theta)^0 + (-\theta)^{-2} + (-\theta)^{-4} + (-\theta)^{-6} + \cdots\right]\gamma_X(0) \\
&\qquad + \left[2(-\theta)^0(-\theta)^{-1} + 2(-\theta)^{-1}(-\theta)^{-2} + 2(-\theta)^{-2}(-\theta)^{-3} + \cdots\right]\gamma_X(1) \\
&= \frac{1}{1-\frac{1}{\theta^2}}(1+\theta^2)\sigma^2 + \frac{2\frac{1}{-\theta}}{1-\frac{1}{\theta^2}}\theta\sigma^2 = \theta^2\sigma^2
\end{aligned}$$

With $h \geq 1$

$$\begin{aligned}
\gamma_W(h) &= \text{cov}(W_{t+h},\, W_t) \\
&= \text{cov}\big((-\theta)^0 X_{t+h} + \cdots + (-\theta)^{-h+1}X_{t+1} + (-\theta)^{-h}X_t + (-\theta)^{-(h+1)}X_{t-1} + \cdots, \\
&\qquad\qquad (-\theta)^0 X_t + (-\theta)^{-1}X_{t-1} + (-\theta)^{-2}X_{t-2} + \cdots\big) \\
&= \left[(-\theta)^{-(h-1)}(-\theta)^0 + (-\theta)^{-h}(-\theta)^{-1} + (-\theta)^{-(h+1)}(-\theta)^{-2} + \cdots\right]\gamma_X(1) \\
&\quad + \left[(-\theta)^{-h}(-\theta)^0 + (-\theta)^{-(h+1)}(-\theta)^{-1} + (-\theta)^{-(h+2)} + (-\theta)^{-2} + \cdots\right]\gamma_X(0) \\
&\quad + \left[(-\theta)^{-(h+1)}(-\theta)^0 + (-\theta)^{-(h+2)}(-\theta)^{-1} + (-\theta)^{-(h+3)}(-\theta)^{-2} + \cdots\right]\gamma_X(1) \\
&= \frac{(-\theta)^{-(h-1)}}{1-\frac{1}{\theta^2}}\theta\sigma^2 + \frac{(-\theta)^{-h}}{1-\frac{1}{\theta^2}}(1+\theta^2)\sigma^2 + \frac{(-\theta)^{-(h+1)}}{1-\frac{1}{\theta^2}}\theta\sigma^2 \\
&= 0
\end{aligned}$$

Thus W_t, $t = 1, 2, 3, \ldots.$, are uncorrelated with $E(W_t) = 0$ and $\text{var}(W_t) = \theta^2\sigma^2$.
Therefore $W_t \sim \text{WN}(0,\, \sigma_W^2)$ with $\sigma_W^2 = \theta^2\sigma^2$.

(ii) Show that $X_t = W_t + \frac{1}{\theta}W_{t-1}$ and it is invertible.

$W_t = \psi_1(B)X_t$ where $\psi_1(B) = \sum_{j=0}^{\infty}(-\theta)^{-j}B^j$. Let $\psi_2(B) = I - \frac{1}{\theta}B$. Then

$$\begin{aligned}\psi_2(B)\psi_1(B) &= [I-(-\theta)^{-1}B]\sum_{j=0}^{\infty}(-\theta)^{-j}B^j = \sum_{j=0}^{\infty}(-\theta)^{-j}B^j + \sum_{j=0}^{\infty} -(-\theta)^{-j-1}B^{j+1} \\ &= I + \sum_{j=1}^{\infty}(-\theta)^{-j}B^j + \sum_{j=1}^{\infty} -(-\theta)^{-j}B^j = I.\end{aligned}$$

Thus $\psi_2(B)W_t = \psi_2(B)\psi_1(B)X_t = IX_t = X_t$, i.e.,

$$X_t = W_t + \frac{1}{\theta}W_{t-1}.$$

But $\psi_2(z) = 0 \Longrightarrow 1 + \frac{z}{\theta} = 0 \Longrightarrow z = -\theta \Longrightarrow |z| = |\theta| > 1$.
So $X_t = W_t + \frac{1}{\theta}W_{t-1}$ is invertible.