Stat775 HW07

1. 3.11 p96
   Show that MA(1) $X_t = Z_t + \theta Z_{t-1}$ with $Z_t \sim \mathrm{WN}(0, \sigma^2)$ has PACV $\alpha(2) = \frac{-\theta^2}{1+\theta+\theta^2}$.

   PACF $\alpha(2)$ is the last component of $\mathbf{\Gamma}_2^{-1}\gamma_2$. With $\gamma(0) = (1+\theta^2)\sigma^2$, $\gamma(1) = \theta\sigma^2$ and $\gamma(h) = 0$ for all $|h| > 1$,

$$\mathbf{\Gamma}_2 = \begin{pmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} \theta\sigma^2 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \text{So } \mathbf{\Gamma}_2^{-1}\gamma_2 &= \begin{pmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{pmatrix}^{-1} \begin{pmatrix} \theta \\ 0 \end{pmatrix} = \tfrac{1}{(1+\theta^2)^2-\theta^2} \begin{pmatrix} 1+\theta^2 & -\theta \\ -\theta & 1+\theta^2 \end{pmatrix} \begin{pmatrix} \theta \\ 0 \end{pmatrix} \\ &= \tfrac{1}{1+\theta^2+\theta^4} \begin{pmatrix} \theta(1+\theta^2) \\ -\theta^2 \end{pmatrix}. \end{aligned}$$

   Thus $\alpha(2) = \frac{-\theta^2}{1+\theta^2+\theta^4}$.

2. 3.12 p96
   With MA(1) for $\mathbf{R}_n\phi_n = \rho_n$ where $\phi_n = \begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n} \end{pmatrix}$, Show $\phi_{n,n-j} = (-\theta)^{-j}(1+\theta^2+\cdots+\theta^{2j})$ and hence PACF $\alpha(n) = \phi_{n,n} = \frac{-(-\theta)^n}{1+\theta^2+\cdots+\theta^{2n}}$.

   With MA(1), the equation $\mathbf{R}_n\phi_n = \rho_n$ is

$$\begin{pmatrix} 1 & \frac{\theta}{1+\theta^2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{\theta}{1+\theta^2} & 1 & \frac{\theta}{1+\theta^2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\theta}{1+\theta^2} & 1 & \frac{\theta}{1+\theta^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\theta}{1+\theta^2} & 1 \end{pmatrix} \begin{pmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \vdots \\ \phi_{n,n-1} \\ \phi_{n,n} \end{pmatrix} = \begin{pmatrix} \frac{\theta}{1+\theta^2} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

   From the nth equation $\frac{\theta}{1+\theta^2}\phi_{n,n-1} + \phi_{n,n} = 0$, $\phi_{n,n-1} = (-\theta)^{-1}(1+\theta^2)\phi_{n,n}$.
   From the $(n-1)$th equation $\frac{\theta}{1+\theta^2}\phi_{n,n-2} + \phi_{n,n-1} + \frac{\theta}{1+\theta^2}\phi_{n,n} = 0$,

$$\phi_{n,n-2} = \frac{1+\theta^2}{\theta}(-\phi_{n,n-1}) - \phi_{n,n} = \frac{1+\theta^2}{\theta}\frac{1+\theta^2}{-(-\theta)}\phi_{n,n} - \phi_{n,n} = (-\theta)^{-2}(1+\theta^2+\theta^4)\phi_{n,n}.$$

   The formula $\phi_{n,n-j} = (-\theta)^{-j}(1+\theta^2+\cdots+\theta^{2j})\phi_{n,n}$ holds for $j=1$ and $j=2$.
   Assume that the formula holds for $j = k-2$ and $j = k-1$. From the $(n-k+1)$th equation,

$$\frac{\theta}{1+\theta^2}\phi_{n,n-k} + \phi_{n,n-(k-1)} + \frac{\theta}{1+\theta^2}\phi_{n,n-(k-2)} = 0,$$

$$\begin{aligned} \phi_{n,n-k} &= \tfrac{1+\theta^2}{-\theta}\phi_{n,n-(k-1)} - \phi_{n,n-(k-2)} \\ &= \left[\tfrac{1+\theta^2}{-\theta}(-\theta)^{-(k-1)}(1+\theta^2+\cdots+\theta^{2k-2}) - (-\theta)^{-(k-2)}(1+\theta^2+\cdots+\theta^{2k-4})\right]\phi_{n,n} \\ &= (-\theta)^{-k}(1+\theta^2+\cdots+\theta^{2k})\phi_{n,n}. \end{aligned}$$

So the formula hods for $j = k$. By induction the formula is true for $j = 1,\, 2, ..., n-1$.
By the first equation, $\phi_{n,n-(n-1)} + \frac{\theta}{1+\theta^2}\phi_{n,n-(n-2)} = \frac{\theta}{1+\theta^2}$,

$$\left[(-\theta)^{-(n-1)}(1+\theta^2+\cdots+\theta^{2n-2}) + \frac{\theta}{1+\theta^2}(-\theta)^{-(n-2)}(1+\theta^2+\cdots+\theta^{2n-4})\right]\phi_{n,n} = \frac{\theta}{1+\theta^2}.$$

Solving the above equation, we have $\phi_{n,n} = \frac{-(-\theta)^n}{1+\theta^2+\cdots+\theta^{2n}}$.