Stat775 KHW06

1. 3.1 on page 94
   Causal? Invertible

   (a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$.

   $0 = 1 + 0.2z - 0.48z^2 \Longrightarrow z = -\frac{5}{4}, \frac{5}{3} \Longrightarrow |z| > 1$. So X_t is causal.

   (b) $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$.

   $0 = \phi(z) = 1 + 1.9z + 0.88z^2 \Longrightarrow z = -\frac{5}{4}, -\frac{10}{11}$ where $|-10/11| < 1$. So X_t is not causal.
   $0 = \theta(z) = 1 + 0.2z + 0.7z^2 \Longrightarrow z = \frac{-1 \pm \sqrt{69}i}{7} \Longrightarrow |z| = \sqrt{\frac{10}{7}} > 1$. X_t is invertible.

   (c) $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$.

   $0 = \phi(z) = 1 + 0.6z \Longrightarrow z = -\frac{5}{3} \Longrightarrow |z| > 1$. So X_t is causal.
   $0 = \theta(z) = 1 + 1.2z \Longrightarrow z = -\frac{5}{6} \Longrightarrow |z| < 1$. So X_t is not invertible.

   (d) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.

   $0 = \phi(z) = 1 + 1.8z + 0.81z^2 \Longrightarrow Z = -\frac{10}{9} \Longrightarrow |z| > 1$. So X_t is causal.

   (e) $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$.

   $0 = \phi(z) = 1 + 1.6z \Longrightarrow z = -\frac{5}{8} \Longrightarrow |z| < 1$. So X_t is not causal.
   $0 = \theta(z) = 1 - 0.4z + 0.04z^2 \Longrightarrow z = 5 \Longrightarrow |Z| > 1$. So X_t is invertible.

2. 3.3 on page 94. For causal X_t in 3.1. Compute ψ_k, $k = 0, 1, 2, 3, 4, 5$ in $X_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k}$.

   (a) $[I - (0.48B - 0.2I)B]X_t = Z_t$.

   $$\begin{aligned} X_t &= [I - (0.48B - 0.2I)B]^{-1} Z_t = \textstyle\sum_{k=0}^{\infty} [(0.48B - 0.2I)B]^k Z_t \\ &= \psi_0 Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \psi_3 Z_{t-3} + \psi_4 Z_{t-4} + \psi_5 Z_{t-5} + \cdots \end{aligned}$$

   where $\psi_0 = 1$; $\psi_1 = -0.2$; $\psi_2 = 0.48 + 0.2^2 = 0.52$
   $\psi_3 = -2 \times 0.2 \times 0.48 - 0.2^3 = -0.20$
   $\psi_4 = 0.48^2 + 3 \times 0.48 \times 0.2^2 + 0.2^4 = 0.2896$
   $\psi_5 = -3 \times 0.48^2 \times 0.2 - 4 \times 0.48 \times 0.2^3 - 0.2^5 = -0.15392$.

   (c) $(I + 0.6B)X_t = (I + 1.2B)Z_t$

   $$\begin{aligned} X_t &= [I - (-0.6B)]^{-1}[I - 2(-0.6B)]Z_t = \textstyle\sum_{k=0}^{\infty} (-0.6B)^k [I - 2(-0.6B)]Z_t \\ &= \left[I + (-0.6B) - (-0.6B)^2 - (-0.6B)^3 - (-0.6B)^4 - (-0.6B)^5 - \cdots\right] Z_t \\ &= \textstyle\sum_{k=0}^{\infty} \psi_k Z_{t-k} \end{aligned}$$

   where $\psi_0 = 1$, $\psi_1 = -0.6$ and $\psi_k = (-1)^{k+1} 0.6^k$ for $k > 1$. Thus
   $\psi_0 = 1$ $\psi_1 = -0.6$
   $\psi_2 = -0.6^2 = -0.36$ $\psi_3 = 0.6^3 = 0.216$
   $\psi_4 = -0.6^4 = -0.1296$ $\psi_5 = 0.6^5 = 0.07776$.

(d) $(I + 1.8B + 0.81B^2)X_t = Z_t$

$$\begin{aligned} X_t &= (I + 1.8B + 0.81B^2)^{-1} Z_t = \textstyle\sum_{k=0}^{\infty} (-1.8B - 0.81B^2)^k Z_t \\ &= \textstyle\sum_{k=0}^{\infty} (-0.9)^k B^k (2I + 0.9B)^k Z_t \\ &= \psi_0 Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \psi_3 Z_{t-3} + \psi_4 Z_{t-4} + \psi_5 Z_{t-5} + \cdots \end{aligned}$$

where

$\psi_0 = 1 \times 0.9^0 = 1$ $\quad$ $\psi_1 = -2 \times 0.9 = -1.8$

$\psi_2 = 3 \times 0.9^2 = 2.43$ $\quad$ $\psi_3 = -4 \times 0.9^3 = -2.916$

$\psi_4 = 5 \times 0.9^4 = 3.2805$ $\quad$ $\psi_5 = -6 \times 0.9^5 = -3.54294.$