L21 Simultaneous confidence intervals

1. Confidence intervals for β_i and confidence region for β

   (1) t-intervals for β_i
   $\widehat{\beta}_i \pm t_{\alpha/2}(n-p)S_{\widehat{\beta}_i}$ is a $1-\alpha$ C.T. for β_i where $S^2_{\widehat{\beta}_i} = MSE\ [(X'X)^{-1}]_{ii}$.

   (2) F-intervals for β_i
   $\widehat{\beta}_i \pm \sqrt{F_\alpha(1,\, n-p)}\ S_{\widehat{\beta}_i}$ is a $1-\alpha$ C.I. for β_i.

   Proof. $t_{\alpha/2}(n-p) = \sqrt{F_\alpha(1,\, n-p)}$.

   Comment: Both $\widehat{\beta}_i$ and $S_{\widehat{\beta}_i}$ are in standard SAS output.

   (3) F-confidence region
   $\left\{\beta \in R^p : (\beta - \widehat{\beta})'[MSE\ (X'X)^{-1}]^{-1}(\beta - \widehat{\beta}) < pF_\alpha(p,\, n-p)\right\}$ is a $1-\alpha$ CR for β.

   Proof. $\frac{(\beta-\widehat{\beta})'(X'X)(\beta-\widehat{\beta})/p}{MSE} < F_\alpha(p,\, n-p)$
   $\iff (\beta - \widehat{\beta})'[MSE\ (X'X)^{-1}]^{-1}(\beta - \widehat{\beta}) < pF_\alpha(p,\, n-p)$.

2. Simultaneous CIs for β_i, $i = 1, ..., k$

   (1) An inequality in probability
   Let A_i, $i = 1, ..., k$, be random events with $P(A_i) \geq 1 - \frac{\alpha}{k}$.
   Then $P(\cap_i A_i) \geq 1 - \alpha$.
   Proof.
   $$\begin{aligned} P(\cap_i A_i) &= 1 - P\left((\cap_i A_i)^c\right) = 1 - P\left(\cup_i A_i^c\right) \\ &\geq 1 - \sum_i P(A_i^c) = 1 - \sum_i [1 - P(A_i)] = 1 - k + \sum_i P(A_i) \\ &\geq 1 - k + \sum_i \left(1 - \frac{\alpha}{k}\right) = 1 - k + k - \alpha = 1 - \alpha. \end{aligned}$$

   (2) Bonferroni simultaneous confidence intervals
   Suppose A_i is a $1 - \frac{\alpha}{k}$ confidence interval for θ_i, $i = 1, ..., k$. Then, $A_1, ..., A_k$ are simultaneous confidence intervals for $\theta_1, ..., \theta_k$ with overall confidence coefficient $1 - \alpha$

   Proof. $P(A_i) \geq 1 - \frac{\alpha}{k}$, $i = 1, ..., k \Longrightarrow P(A_1 \cap \cdots \cap A_k) \geq 1 - \alpha$.

Ex1: For $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$, construct simultaneous CIs for β_i, $i = 0, 1, 2, 3$ with overall confidence coefficient 90%

$1 - \alpha = 0.90 \Longrightarrow \alpha = 0.1 \Longrightarrow \frac{\alpha}{4} = 0.025 \Longrightarrow 1 - \frac{\alpha}{4} = 0.975$.
So one can construct CIs one by one, each one has CC 97.5%, i.e.,
$\widehat{\beta}_i \pm t_{0.0125}(n-p)S_{\widehat{\beta}_i}$, $i = 0, 1, 2, 3$. `proc reg; model y=x1 x2 x3/clb alpha=0.025;`

Ex2: `proc reg; model y=x1 x2 x3/clb alpha=0.05; run;`
produces 95% CIs for β_i, $i = 0, 1, 2, 3$.
Pick two as simultaneous CIs. Find the overall confidence coefficient.

$\frac{\alpha}{2} = 0.05 \Longrightarrow \alpha = 0.10 \Longrightarrow 1 - \alpha = 0.90$.
So the overall confidence coefficient is 90%

3. Scheffe's intervals

   (1) Extended Cauchy-Schwartz inequality
   For $u,\, v \in R^p$, $(u'v)^2 \le (u'u)(v'v)$ is known as Cauchy-Schwartz inequality.
   With positive definite B, let $u = B^{1/2}x$ and $v = B^{-1/2}y$. Then

$$(x'y)^2 \le (x'Bx)(y'B^{-1}y).$$

   This is extended Cauchy-Schwartz inequality. Thus with given $c > 0$

$$y'B^{-1}y < c \Longrightarrow \frac{(x'y)^2}{x'Bx} < c.$$

   (2) Relation of events
   With $x = e_i$, $y = \beta - \widehat{\beta}$, $B = MSE\,(X'X)^{-1}$ and $c = pF_\alpha(p,\, n-p)$,
   $A = \left\{y'B^{-1}y < c\right\}$ and $A_i = \left\{\frac{(x'y)^2}{x'Bx} < c\right\}$, $i = 1,\, 2, ...$ are random events and

$$A \subset A_i \text{ for all } i \Longrightarrow A \subset \cap_i A_i \Longrightarrow P(A) \le P(\cap_i A_i).$$

   (3) Scheffe's intervals for β_i for $i = 1, 2, ...$
   Note that $A = \left\{y'B^{-1}y\right\} = \left\{(\beta - \widehat{\beta})'[MSE\ (X'X)^{-1}]^{-1}(\beta - \widehat{\beta}) < pF_\alpha(p,\, n-p)\right\}$ with $P(A) = 1 - \alpha$, and

$$\begin{aligned} A_i &= \left\{\frac{(x'y)^2}{x'Bx} < c\right\} = \left\{\frac{[e_i'(\beta-\widehat{\beta})]^2}{e_i' MSE\,(X'X)^{-1}e_i} < pF_\alpha(p,\, n-p)\right\} \\ &= \left\{\left(\frac{(\beta_i-\widehat{\beta}_i)^2}{s_{\widehat{\beta}_i}}\right)^2 < pF_\alpha(p,\, n-p)\right\} = \left\{\left(\beta_i - \widehat{\beta}_i\right)^2 < pF_\alpha(p,\, n-p)S_{\widehat{\beta}_i}\right\} \\ &= \left\{\beta_i \in \widehat{\beta}_i \pm \sqrt{pF_\alpha(p,\, n-p)}\, s_{\widehat{\beta}_i}\right\}. \end{aligned}$$

   Thus $P(\cap_i A_i) \ge P(A) \ge 1 - \alpha$ implies that $\beta_i \in \widehat{\beta}_i \pm \sqrt{pF_\alpha(p,\, n-p)}\, S_{\widehat{\beta}_i}$, $i = 1,\, 2, ...,$ are simultaneous CIs for β_i, $i = 1,\, 2, ...$ with overall cc $1 - \alpha$. These intervals are known as Scheffe's intervals.

Ex3: Comparison
By Bonferroni method k simultaneous CIs for β_i, $i = 1, .., k$ with overall confidence coefficient $1 - \alpha$ are

$$\beta_i \in \widehat{\beta}_i \pm \sqrt{F_{\alpha/k}(1,\, n-p)}\, S_{\widehat{\beta}_i},\ i = 1, ..., k$$

The Scheffe's interval for β_i, $i = 1, ..., k$ with overall confidence coefficient $1 - \alpha$ are

$$\beta_i \in \widehat{\beta}_i \pm \sqrt{pF_\alpha(p,\, n-p)}\, S_{\widehat{\beta}_i},\ \imath = 1, ..., k$$

Comment: Scheffe's intervals are wider. Bonferroni intervals are k specific. But Scheffe's intervals are for all $k = 1,\, 2, ...$