L14 t-tests in regression

1. t-tests by rejection regions

   (1) Framework

   If $\frac{\widehat{\theta}-\theta}{S} \sim t(df)$, then

$H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ Reject H_0 if $t > t_\alpha(df)$

$H_0: \theta \geq \theta_0$ vs $H_a: \theta < \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ Reject H_0 if $t < -t_\alpha(df)$

$H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ Reject H_0 if $t < -t_{\alpha/2}(df)$ or $t > t_{\alpha/2}(df)$

   (2) Comments

   (i) In the definition of test statistic $t = \frac{\widehat{\theta}-\theta_0}{S}$, θ_0 is from the hypotheses and $\theta_0 \in H_0$.

   (ii) Lower-sided/upper-sided/two-sided H_a

   $\Longleftrightarrow$ Lower-sided/upper-sided/two-sided rejection region

   (iii) $P(t(df) \in \text{Rejection region}) = \alpha$.

   (3) Ex1: Presentation of your test

   For $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x + \beta_4 x_4 + \epsilon$, is $E(y)$ an increasing function of x_4? Support your conclusion by a test at level 0.05. $n = 24$

$H_0: \beta_4 \leq 0$ vs $H_a: \beta_0 > 0$ Test statistic: $t = \frac{\widehat{\beta}_4}{S_{\widehat{\beta}_4}}$ Reject H_0 if $t > 1.729$ for $\alpha = 0.05$ $t_{ob} = 0.47$ Fail to reject H_0 Data do not show evidence to support that claim

2. t-tests by p-values

   (1) Framework

   If $\frac{\widehat{\theta}-\theta}{S} \sim t(df)$, then

$H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ p-value: $P(t(df) > t_{0b})$

$H_0: \theta \geq \theta_0$ vs $H_a: \theta < \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ p-value: $P(t(df) < t_{ob})$

$H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S}$ p-value: $2P(t(df) > \|t_{0b}\|)$

(2) Comments

(i) In the definition of test statistic $t = \frac{\widehat{\theta}-\theta_0}{S}$, θ_0 is from the hypotheses and $\theta_0 \in H_0$.

(ii) Lower-sided/upper-sided/two-sided H_a have

Lower-sided/upper-sided/two-sided tails cut off by t_{ob} as p-values

(3) Ex2: Presentation of your test

Do Ex1 using p-values

$H_0 : \beta_4 \leq 0$ vs $H_a : \beta_0 > 0$
Test statistic: $t = \frac{\widehat{\beta}_4}{S_{\widehat{\beta}_4}}$
p-value: $P(t(19) > t_{ob})$
$t_{ob} = 0.47$, p-value: $\frac{1}{2} \times 0.64 = 0.32$
Fail to reject H_0
Data do not show evidence to support that claim

3. Relations of t-test and t-intervals

(1) α-level tests and $1-\alpha$ CIs

Fail to reject $H_0 : \theta \leq \theta_0$ at $\alpha \iff \theta_0$ is in $1-\alpha$ upper-sided CI for θ

Fail to reject $H_0 : \theta \geq \theta_0$ at $\alpha \iff \theta_0$ is in $1-\alpha$ lower-sided CI for θ

Fail to reject $H_0 : \theta = \theta_0$ at $\alpha \iff \theta_0$ is in $1-\alpha$ two-sided CI for θ

Proof. Show the first one.

Fail to reject $H_0 : \theta \leq \theta_0$ at $\alpha \iff t = \frac{\widehat{\theta}-\theta_0}{S} < t_\alpha(df) \iff \theta_0 > \widehat{\theta} - t_\alpha(df)S$

$\iff \theta_0 \in (\widehat{\theta} - t_\alpha(df)S, \infty) \iff \theta_0$ is in $1-\alpha$ upper-sided CI for θ

(2) p-value and smallest confidence coefficient

Test on $H_0 : \theta \leq \theta_0$ produced p-value $p \iff$ Fail to reject $H_0 : \theta \leq \theta_0$ at $\alpha < p$

$\iff \theta_0$ is in upper-sided CI for θ with cc $1-\alpha > 1-p$

Test on $H_0 : \theta \geq \theta_0$ produced p-value $p \iff$ Fail to reject $H_0 : \theta \leq \theta_0$ at $\alpha < p$

$\iff \theta_0$ is in lower-sided CI for θ with cc $1-\alpha > 1-p$

Test on $H_0 : \theta = \theta_0$ produced p-value $p \iff$ Fail to reject $H_0 : \theta = \theta_0$ at $\alpha < p$

$\iff \theta_0$ is in two-sided CI for θ with cc $1-\alpha > 1-p$.

Interval with smallest cc has smallest width.

Ex3: In Ex1, we fail to reject $H_0 : \beta_4 \leq 0$ at $\alpha = 0.05$.

So 0 is in a 95% upper-sided CI for β_4.

Ex4: In Ex2, test on $H_0 : \beta_4 \leq 0$ produced p-value: 0.32.

So 0 is in upper-sided CI for β_4 with smallest confidence coefficient 68%.

Ex5: In Ex2, the p-value for the test on $H_0 : \beta_4 = 0$ is 0.64.

So 0 is in CI for β_4 with smallest confidence coefficient 36%.