L05 t-tests

1. t-tests

   Suppose θ is estimated by $\widehat{\theta}$ and $\frac{\widehat{\theta}-\theta}{S_{\widehat{\theta}}} \sim t(k)$.

   (1) by rejection region

$H_0: \theta \geq \theta_0$ vs $H_a: \theta < \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ Reject H_0 if $t < -t_\alpha(k)$	$H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ Reject H_0 if $t > t_\alpha(k)$

$H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ Reject H_0 if $t < -t_{\alpha/2}(k)$ or $t > t_{\alpha/2}(k)$

   are lower-sided alternative, upper-sided alternative and two-sided alternative α-level test

   Proof. Consider the first lower-sided alternative test with lower-sided rejection region.

$$\text{Note that under } H_0: \theta \geq \theta_0 \iff \frac{\widehat{\theta}-\theta}{S_{\widehat{\theta}}} \leq \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$$
$$\iff P\left(\frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}} < c\right) \leq P\left(\frac{\widehat{\theta}-\theta}{S_{\widehat{\theta}}} \leq c\right) \text{ for all } c.$$

$$\begin{aligned}\text{So, } P\left(\text{Type I error}\right) &= P\left(\text{Rejecting } H_0 | H_0 \text{ is true}\right) = P\left(\frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}} < -t_\alpha(k) | H_0\right) \\ &\leq P\left(\frac{\widehat{\theta}-\theta}{S_{\widehat{\theta}}} < -t_\alpha(k)\right) = P\left(t(k) \leq -t_\alpha(k)\right) = \alpha.\end{aligned}$$

   Thus the lower-sided alternative test is an α-level test.

   Comment: The lower-sided alternative test has lower-sided rejection region,
   The upper-sided alternative test has upper-sided rejection region
   and the two-sided alternative test has two-sided rejection region.

   (2) by p-value

$H_0: \theta \geq \theta_0$ vs $H_a: \theta < \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ p-value: $P\left(t(k) < t_{ob}\right)$	$H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ p-value: $P\left(t(k) > t_{ob}\right)$

$H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ Test statistic: $t = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$ p-value: $2P\left(t(k) > \lvert t_{ob}\rvert\right)$

   are the tests equivalent to that in (1).

   Proof. Consider the second upper-sided alternative test.

$$\begin{aligned} & H_0 \text{ is rejected by using } \alpha\text{-level rejection region} \\ \iff & t_{ob} > t_\alpha(k) \iff P\left(t(k) > t_{ob}\right) < P(t(k) > t_\alpha(k)) = \alpha \iff \text{p-value} < \alpha \\ \iff & H_0 \text{ is rejected by using p-value.}\end{aligned}$$

   Comment: Lower-sided alternative test has lower-sided probability as p-value.
   Upper-sided alternative test has upper-sided probability as p-value.
   Two-sided alternative test has two sided probability as p-value since

$$2P\left(t(k) > |t_{ob}|\right) = P\left(t(k) < -|t_{ob}|\right) + P\left(t(k) > |t_{ob}|\right).$$

2. t-tests in $y = \beta_0 + \beta_1 x + \epsilon$

(1) Practical needs
After confirming the usefulness of the model, there may be still many questions need to be answered. For example
Is $E(y(x))$ and increasing function of x?$\Longleftrightarrow$ Is $H_a: \beta_1 > 0$ true?
$E(y(0)) = 5$? $\Longleftrightarrow H_0: \beta_0 = 5$ true?
$E(y(3)) \geq 10$? $\Longleftrightarrow H_0: E(y(3)) \geq 10$ true?

(2) Framework
By the framework in 1 and $\frac{\widehat{\beta}_0 - \beta_0}{S_{\widehat{\beta}_0}} \sim t(n-2)$, $\frac{\widehat{\beta}_1 - \beta_1}{S_{\widehat{\beta}_1}} \sim t(n-2)$, $\frac{\widehat{y}(x_0) - E(y(x_0))}{S_{\widehat{y}(x_0)}} \sim t(n-2)$, there are three types tests on each of β_0, β_1 and $E(y(x_0)) = \beta_0 + \beta_1 x_0$ by rejection region and by p-value.

(3) To carry out the tests $\widehat{\beta}_0$, $S_{\widehat{\beta}_0}$; $\widehat{\beta}_1$, $S_{\widehat{\beta}_1}$; and $\widehat{y}(x_0)$, $S_{\widehat{y}(x_0)}$ are needed.

Ex1: Based on a sample of size $n = 20$ one obtains $\widehat{\beta}_1 = -37$ and $S_{\widehat{\beta}_1} = 3$. Is the regression function a decreasing function of x? Support your conclusion by a test at level 0.05.

From App $t_{0.05}(18) = 1.734$.

$H_0: \beta_1 \geq 0$ vs $H_a: \beta_1 < 0$
Test statistic $t = \frac{\widehat{\beta}_1}{S_{\widehat{\beta}_1}}$
Rejet H_0 if $t < -1.734$ for $\alpha = 0.05$
$t_{ob} = \frac{-37}{3} = -12$
Reject H_0. $E(y(x))$ is a decreasing function of x

Comment: t-test and F-test on $H_0: \beta_1 = 0$ are equivalent, i.e., they produce the same p-value, and reach the same conclusion.

3. t-tests in $y = \beta x + \epsilon$

(1) By the framework in 1 and $\frac{\widehat{\beta} - \beta}{S_{\widehat{\beta}}} \sim t(n-1)$ and $\frac{\widehat{y}(x_0) - E(y)(x_0)}{S_{\widehat{y}(x_0)}} \sim t(n-1)$ there are three types of tests on each of β and $E(y(x_0))$ by rejection regions or by p-values.

(2) To carry out the tests $\widehat{\beta}$ and $S_{\widehat{\beta}}$; $\widehat{y}(x_0)$ and $S_{\widehat{y}(x_0)}$ are needed.

Ex2: A sample for the model without intercept produced

$$n = 4, \sum x = 2, \sum y = 14, \sum x^2 = 6, \sum y^2 = 54, \sum xy = 4.$$

For future $x_0 = 2$ test $H_0: E(y(2)) = 3$.

Computing $MSE = \frac{1}{n-1}\left[\sum y^2 - \frac{(\sum xy)^2}{\sum x^2}\right] = 17.1111$, $\widehat{\beta} = \frac{\sum xy}{\sum x^2} = 0.6667$
$\widehat{y}(2) = \widehat{\beta} \times 2 = 1.333$, $S^2_{\widehat{y}(2)} = MSE\frac{2^2}{\sum x^2} = 3.38^2$

$H_0: E(y(2)) = 3$ versus $H_a: E(y(2)) \neq 3$
Test Statistic: $t = \frac{\widehat{y}(2) - 3}{S_{\widehat{y}(2)}}$
p-value: $2P\left(t(n-1) > \lvert t_{ob} \rvert\right)$
$t_{pb} = \frac{1.333 - 3}{3.38} = -0.49$, p-value
p-value: $2P(t(3) > 0.49) = 2 \times 0.329 = 0.658$
Fail to reject H_0 at the level 0.65
Data do not show evidence against the claim that $E(y(2)) = 3$

L06: t-intervals

1. t-intervals and their relations to t-tests
   Suppose θ is estimated by $\widehat{\theta}$ and $\frac{\widehat{\theta}-\theta}{s_{\widehat{\theta}}} \sim t(k)$.

   (1) t-confidence intervals for θ with confidence coefficient $1-\alpha$

(i) Two-sided CI:	$\widehat{\theta} \pm t_{\alpha/2}(k)s_{\widehat{\theta}}$	is a $1-\alpha$ two-sided CI for θ.
(ii) Lower-sided CI:	$\left(-\infty, \widehat{\theta} + t_{\alpha}(k)s_{\widehat{\theta}}\right)$	is a $1-\alpha$ lower-sided CI for θ.
(iii) Upper-sided CI:	$\left(\widehat{\theta} - t_{\alpha}(k)s_{\widehat{\theta}}, \infty\right)$	is a $1-\alpha$ upper-sided CI for θ.

   Proof. (i) Need to show $P\left(\widehat{\theta} - t_{\alpha/2}(k)s_{\widehat{\theta}} < \theta < \widehat{\theta} + t_{\alpha/2}(k)s_{\widehat{\theta}}\right) = 1-\alpha$.

   $$\begin{aligned} 1-\alpha &= P(-t_{\alpha/2}(k) < t(k) < t_{\alpha/2}(k)) = P\left(-t_{\alpha/2}(k) < \tfrac{\widehat{\theta}-\theta}{s_{\widehat{\theta}}} < t_{\alpha/2}(k)\right) \\ &= P\left(\widehat{\theta} - t_{\alpha/2}(k)s_{\widehat{\theta}} < \theta < \widehat{\theta} + t_{\alpha/2}(k)s_{\widehat{\theta}}\right). \end{aligned}$$

   (ii) Need to show $P\left(-\infty < \theta < \widehat{\theta} + t_{\alpha}(k)s_{\widehat{\theta}}\right) = 1-\alpha$.

   $$\begin{aligned} 1-\alpha &= P(-t_{\alpha}(k) < t(k) < \infty) = P\left(-t_{\alpha}(k) < \tfrac{\widehat{\theta}-\theta}{s_{\widehat{\theta}}} < \infty\right) \\ &= P\left(-\infty < \theta < \widehat{\theta} + t_{\alpha}(k)s_{\widehat{\theta}}\right). \end{aligned}$$

   (iii) Skipped

   (2) Relations of $1-\alpha$ CI and α-level test

(i) θ_0 is in $1-\alpha$ two-sided CI	$\Longleftrightarrow$	α-level test on $H_0 : \theta = \theta_0$ fails to reject H_0
(ii) θ_0 is in $1-\alpha$ lower-sided CI	$\Longleftrightarrow$	α-level test on $H_0 : \theta \geq \theta_0$ fails to reject H_0
(iii) θ_0 is in $1-\alpha$ upper-sided CI	$\Longleftrightarrow$	α-level test on $H_0 : \theta \leq \theta_0$ fails to reject H_0

   Proof. (i)
   $$\begin{aligned} & \theta_0 \text{ is in } 1-\alpha \text{ two-sided confidence interval for } \theta \\ \Longleftrightarrow\ & \theta_0 \in \widehat{\theta} \pm t_{\alpha/2}(k)S_{\widehat{\theta}} \longleftrightarrow -t_{\alpha/2}(k) < \tfrac{\widehat{\theta}-\theta}{S_{\widehat{\theta}}} < t_{\alpha/2}(k) \\ \Longleftrightarrow\ & \alpha\text{-level test on } H_0 : \theta = \theta_0 \text{ fails to reject } H_0 \end{aligned}$$

   (ii)
   $$\begin{aligned} & \theta_0 \text{ is in } 1-\alpha \text{ lower-sided CI for } \theta \\ \Longleftrightarrow\ & -\infty < \theta_0 < \widehat{\theta} + t_{\alpha}(k)S_{\widehat{\theta}} \Longleftrightarrow \tfrac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}} \\ \Longleftrightarrow\ & \alpha\text{-level test on } H_0 : \theta \geq \theta_0 \text{ fails to reject } H_0 \end{aligned}$$

   (iii) Skipped

2. t-intervals in $y = \beta_0 + \beta_1 x + \epsilon$

   (1) Practical needs
   After F-test confirms the usefulness of the model, many estimation problems arise.
   For example, when x increases by 1, we may need to estimate the increment in $E(y)$ by a confidence interval, i.e., a confidence interval for β_1.
   We always want to have an interval estimate for the mean of a future response when $x = x_0$, a confidence interval for $E(y(x_0)) = \beta_0 + \beta_1 x_0$.

(2) Framework
Based on the framework in 1 and

$$\frac{\widehat{\beta}_0-\beta_0}{S_{\widehat{\beta}_0}} \sim t(n-2),\ \frac{\widehat{\beta}_1-\beta_1}{S_{\widehat{\beta}_1}} \sim t(n-2) \text{ and } \frac{\widehat{y}(x_0)-E(y(x_0))}{S_{\widehat{y}(x_0)}} \sim t(n-2),$$

there are three types confidence intervals for β_0, β_1 and $E(y(x_0))$. For example

$$\begin{array}{ll} \left(-\infty,\ \widehat{\beta}_0+t_\alpha(n-2)S_{\widehat{\beta}_0}\right) & \text{is a } 1-\alpha \text{ lower-sided CI for } \beta_0 \\ \left(\widehat{\beta}_1-t_\alpha(n-2)S_{\widehat{\beta}_1},\ \infty\right) & \text{is a } 1-\alpha \text{ upper-sided CI for } \beta_1 \\ \widehat{y}(x_0)\pm t_{\alpha/2}(n-2)S_{\widehat{y}(x_0)} & \text{is a } 1-\alpha \text{ two-sided CI for } E(y(x_0)) \end{array}$$

Ex1: Based on a sample of size $n=20$ one obtains $\widehat{\beta}_1=-37$ and $S_{\widehat{\beta}_1}=3$. Find a lower-sided confidence coefficient for β_1 with confidence coefficient 0.95.
$1-\alpha=0.95 \Longrightarrow \alpha=0.05 \Longrightarrow t_\alpha(n-2)=1.734$
$\left(-\infty,\ \widehat{\beta}_1+t_\alpha(n-2)S_{\widehat{\beta}_1}\right)=(-\infty,\ -37+1.734\times 3)=(-\infty,\ -31.798)$
is a 95% CI for β_1.

(3) Relations to α-level tests
The relations of α-level tests and $1-\alpha$ CI hold. For example

$$\begin{array}{ll} \text{Ex1 in L05} & H_0:\ \beta_1\geq 0 \text{ is rejected at the level } \alpha=0.05 \\ \Longleftrightarrow & 0 \text{ is not in the 95\% lower-sided CI for } \beta_1 \Longleftrightarrow \text{Ex1 L06} \end{array}$$

3. t-intervals in $y=\beta x+\epsilon$

(1) Confidence intervals
By the framework in 1 and

$$\frac{\widehat{\beta}-\beta}{S_{\widehat{\beta}}} \sim t(n-1) \text{ and } \frac{\widehat{y}(x_0)-E(y(x_0))}{S_{\widehat{y}(x_0)}} \sim t(n-1)$$

there are lower-sided, upper-sided and two-sided confidence intervals for β and for $E(y(x_0))=\beta x_0$

Ex2: A sample for the model without intercept produced

$$n=4,\ \sum x=2,\ \sum y=14,\ \sum x^2=6,\ \sum y^2=54,\ \sum xy=4.$$

For a 95% confidence interval for $E(y(2))$.
$\widehat{y}(2)=\widehat{\beta}\times 2=1.333$, $S^2_{\widehat{y}(3)}=3.38^2$
$t_{\alpha/2}(n-1)=t_{0.025}(3)=3.182$

$\widehat{y}(2)\pm t_{\alpha/2}(n-1)S_{\widehat{y}(2)}=1.333\pm 3.182\times 3.38=1.333\pm 10.755=(-9.422,\ 12.088)$
is a 95% CI for $E(y(2))$.

(2) The relation of α-level tests and $1-\alpha$ CI holds
For example Ex2 in L05 shows p-value=0.65 for test on $H_0:\ E(y(2))=3$. So H_0 is rejected at $\alpha=0.05$. Hence 3 is in 95% CI for $E(y(2))$. This is the conclusion of Ex2 here in L06. What if $\alpha=0.15,\ 0.25,\ 0.45$?