L24 Vectorizing matrix product and commutation matrices

1. Vectorizing matrix product

   (1) Tools
   Recall: For $x \in C^m$ and $y \in C^n$

$$\begin{array}{c} x \otimes y' = y' \otimes x = xy' \text{ and } x \otimes y^* = y^* \otimes x = xy^* \\ \mathrm{vec}(x \otimes y') = \mathrm{vec}(y' \otimes x) = \mathrm{vec}(xy') = y \otimes x \\ \text{and } \mathrm{vec}(x \otimes y^*) = \mathrm{vec}(y^* \otimes x) = \mathrm{vec}(xy^*) = \overline{y} \otimes x. \end{array}$$

   (2) Vectorizing matrix product

$$\mathrm{vec}(AXB) = (B' \otimes A)\,\mathrm{vec}(X)$$

   Proof Suppose $X \in R^{p\times q}$ and $I_q = (e_1, ..., e_q)$.

$$\begin{array}{rcl} \mathrm{vec}(AXB) & = & \mathrm{vec}[A(X_1, .., X_q)(e_1, .., e_q)'B] = \mathrm{vec}\left[(AX_1, ..., AX_q)\begin{pmatrix} e_1'B \\ \vdots \\ e_q'B \end{pmatrix}\right] \\ & = & \mathrm{vec}[(AX_1)(B'e_1)' + \cdots + (AX_q)(B'e_q)'] \\ & = & \mathrm{vec}[(AX_1)(B'e_1)'] + \cdots + \mathrm{vec}[(AX_q)(B'e_q)'] \\ & = & (B'e_1) \otimes (AX_1) + \cdots + (B'e_q) \otimes (AX_q) \\ & = & (B' \otimes A)(e_1 \otimes X_1) + \cdots + (B' \otimes A)(e_q \otimes X_q) \\ & = & (B' \otimes A)\,\mathrm{vec}(X_1e_1') + \cdots + (B' \otimes A)\,\mathrm{vec}(X_1e_q') \\ & = & (B' \otimes A)\,\mathrm{vec}(X_1e_1' + \cdots + X_qe_q') \\ & = & (B' \otimes A)\mathrm{vec}(XI') = (B' \otimes A)\,\mathrm{vec}(X). \end{array}$$

   Ex1: For $A \in C^{m\times n}$ and $B \in C^{n\times p}$
   $\mathrm{vec}(AB) = \mathrm{vec}(AI_nB) = (B' \otimes A)\mathrm{vec}(I_n)$; $\mathrm{vec}(AB) = \mathrm{vec}(I_mAB) = (B' \otimes I_m)\,\mathrm{vec}(A)$;
   $\mathrm{vec}(AB) = \mathrm{vec}(ABI_p) = (I_p \otimes A)\mathrm{vect}(B)$.

   (3) Comment
   $Y = AXB$ is a linear transformation from $C^{p\times q}$ to $C^{m\times n}$. $C^{p\times q}$ and $C^{m\times n}$ are isomorphic to C^{pq} and C^{mn} with isomorphism $\mathrm{vec}(\cdot)$. Would $\mathrm{vec}(X) \to \mathrm{vec}(Y)$ be a linear transformation? If so, then $\mathrm{vec}(Y) = D\,\mathrm{vec}(X)$. Can we find matrix D? Those questions are answered.

2. Commutation matrices

   (1) Definition of $K_{m,n}$
   With $I_m = (e_{1,m}, ..., e_{m,m})$ and $I_n = (e_{1,n}, ..., e_{n,n})$ define commutation matrix

$$K_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} e_{i,m} \otimes e_{j,n}' \otimes e_{i,m}' \otimes e_{j,n} \in R^{mn\times mn}.$$

   Comment: $K_{m,n}$ is real, square and produced by I_m and I_n.

   (2) $K_{m,n}' = K_{n,m}$.

$$\begin{array}{rcl} \textbf{Proof } K_{m,n}' & = & \left[\sum_{i=1}^{m}\sum_{j=1}^{n} e_{i,m} \otimes e_{j,n}' \otimes e_{i,m}' \otimes e_{j,n}\right]' \\ & = & \sum_{i=1}^{m}\sum_{j=1}^{n} e_{i,m}' \otimes e_{j,n} \otimes e_{i,m} \otimes e_{j,n}' \\ & = & \sum_{j=1}^{n}\sum_{i=1}^{m} e_{j,n} \otimes e_{i,m}' \otimes e_{j,n}' \otimes e_{i,m} = K_{n,m} \end{array}$$

   Comment: Transposed $K_{m,n}$ is a commutation matrix produced by I_n and I_m.

(3) $K_{m,n}^{-1} = K_{n,m}$.

Proof $\quad K_{m,n}K_{n,m}$

$$
\begin{aligned}
&= \left[\sum_i^m \sum_j^n (e_{i,m} \otimes e'_{j,n} \otimes e'_{i,m} \otimes e_{j,n})\right] \left[\sum_p^n \sum_q^m (e_{p,n} \otimes e'_{q,m} \otimes e'_{p,n} \otimes e_{q,m})\right] \\
&= \sum_i^m \sum_j^n \sum_p^n \sum_q^m (e_{i,m} \otimes e'_{j,n} \otimes e'_{i,m} \otimes e_{j,n})(e'_{q,m} \otimes e_{p,n} \otimes e_{q,m} \otimes e'_{p,n}) \\
&= \sum_i^m \sum_j^n \sum_p^n \sum_q^m (e_{i,m}e'_{q,m}) \otimes (e'_{j,n}e_{p,n}) \otimes (e'_{i,m}e_{q,m}) \otimes (e_{j,n}e'_{p,n}) \\
&= \sum_i^m \sum_j^n (e_{i,m}e'_{i,m}) \otimes (e_{j,n}e'_{j,n}) = \sum_i^m (e_{i,m}e'_{i,m}) \otimes \sum_j^n (e_{j,n}e'_{j,n}) \\
&= I_m \otimes I_n = I_{mn}
\end{aligned}
$$

Comment: $K_{m,n}$ is an orthogonal matrix.

Ex2: $K_{m,1} = \sum_{i=1}^m e_{i,m} \otimes e'_{i,m} = \sum_{i=1}^m e_{i,m}e'_{i,m} = I_m$.
$K_{1,n} = \sum_{j=1}^n e'_{j,n} \otimes e_{j,n} = \sum_{j=1}^n e_{j,n}e'_{j,n} = I_n$. So $K_{m,1} = I_m = K_{1,m}$.

3. An applications of commutation matrices

(1) Commutation matrix in vectorization
For $A \in C^{m\times n}$ $\quad K_{m,n}\,\mathrm{vec}(A) = \mathrm{vec}(A')$.

Proof For $A = (a_{ij})_{m\times n}$,

$$A' = \sum_{i=1}^m \sum_{j=1}^n (e'_{i,m}Ae_{j,n})(e_{j,n}e'_{i,m}) = \sum_{i=1}^m \sum_{j=1}^n (e_{j,n}e'_{i,m})A(e_{j,n}e'_{i,m}).$$

$$
\begin{aligned}
\text{So } \mathrm{vec}(A') &= \sum_{i=1}^m \sum_{j=1}^n \mathrm{vec}\left[(e_{j,n}e'_{i,m})A(e_{j,n}e'_{i,m})\right] \\
&= \sum_{i=1}^m \sum_{j=1}^n \left[(e_{j,n}e'_{i,m})' \otimes (e_{j,n}e'_{i,m})\right] \mathrm{vec}(A) = K_{m,n}\,\mathrm{vec}(A).
\end{aligned}
$$

(2) Commutation matrices in Kronecker product
For $A \in C^{m\times n}$ and $B \in C^{p\times q}$, $K_{p,m}\,(A \otimes B)\,K_{n,q} = B \otimes A$.

Proof For all $X \in C^{q\times n}$,

$$
\begin{aligned}
K_{p,m}(A \otimes B)\,\mathrm{vec}(X) &= K_{p,m}\mathrm{vec}(BXA') = \mathrm{vec}(AX'B') \\
&= (B \otimes A)\,\mathrm{vec}(X') = (B \otimes A)\,K_{q,n}\mathrm{vec}(X).
\end{aligned}
$$

So $K_{p,m}(A \otimes B) = (B \otimes A)\,K_{q,n}$. Thus $K_{q,m}(A \otimes B)K_{n,q} = B \otimes A$.

Comment: For commutation of A and B in Kronecker product, one needs to use matrices $K_{m,n}$. These matrices are therefore called commutation matrices.

Ex3: With $A \in C^{m\times n}$ and $y \in C^p$, $K_{p,m}(A \otimes y)K_{n,1} = y \otimes A$, i.e., $K_{p,m}(A \otimes y) = y \otimes A$.