L22: Vectorization and Kronecker product

1. Matrix vectorization

   (1) Definition

   For $X = (X_1, ..., X_n) \in C^{m\times n}$, define $\text{vec}(X) = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in C^{mn}$.

   (i) $\forall\, X \in C^{m\times n}$, $\exists !\, Y \in C^{mn}$ such that $\text{vec}(X) = Y$, i.e., $\text{vec}(\cdot)$ is an injection from $C^{m\times n}$ to C^{mn}. (picture)

   (ii) $\forall\, Y \in C^{mn}$ $\exists\, X \in C^{m\times n}$ such that $\text{vec}(X) = Y$, i.e., $\text{vec}(\cdot)$ is a surjection from $C^{m\times n}$ to C^{mn}.

   (iii) $\forall\, Y \in C^{mn}$, $\exists !\, X \in C^{m\times n}$ such that $\text{vec}(X) = Y$.
   (i)+(iii) means that $\text{vec}(\cdot)$ is a 1-1 mapping between $C^{m\times n}$ and C^{mn}.

   (2) $\text{vec}(\cdot)$ is a linear transformation
   $\text{vec}(\alpha A + \beta B) = \alpha\ \text{vec}(A) + \beta\ \text{vec}(B)$.

   (3) $\text{vec}(\cdot)$ preserves Frobenius inner product: $\langle A,\, B\rangle = \langle \text{vec}(A),\, \text{vec}(B)\rangle$.
   For $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$,

   $$\langle A,\, B\rangle = \text{tr}(B^*A) = \text{tr}(AB^*) = \text{tr}(\overline{B}A') = \text{tr}(A'\overline{B}) = \textstyle\sum_i \sum_j a_{ij}\bar{b}_{ij}.$$
   $$\langle \text{vec}(A),\, \text{vec}(B)\rangle = [\text{vec}(B)]^*\text{vec}(A) = [\text{vec}(A)]'\overline{[\text{vec}(B)]} = \textstyle\sum_i \sum_j a_{ij}\bar{b}_{ij}.$$

   For real matrices

   $$\langle A,\, B\rangle = \text{tr}(B'A) = \text{tr}(AB') = \text{tr}(A'B) = \text{tr}(BA') = \textstyle\sum_i \sum_j a_{ij}b_{ij}$$
   $$\langle \text{vec}(A),\, \text{vec}(B)\rangle = [\text{vec}(B)]'\text{vec}(A) = [\text{vec}(A)]'\text{vec}(B) = \textstyle\sum_i \sum_j a_{ij}b_{ij}.$$

   Comment: $C^{m\times n}$ and C^{mn} as two sets, as two linear spaces and as two linear spaces with inner products are isomorphic (the same). The 1-1 mapping $\text{vec}(\cdot)$ is called an isomorphism.

2. Kronecker product

   (1) Definition

   For $A = (a_{ij})_{m\times n}$ and $B \in C^{p\times q}$, $A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in C^{(mp)\times(nq)}$ is the Kronecker product of A and B.

   (2) Basic properties

   (i) Kronecker product is not commutative

   Ex1: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x \otimes y = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$, $y \otimes x = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$. $x \otimes y \neq y \otimes x$

   (ii) Kronecker product is associative
   $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. So we simply write $A \otimes B \otimes C$.

   (iii) Distribution law holds
   $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$ and $C \otimes (A + B) = (C \otimes A) + (C \otimes B)$

   (iv) Kronecker product with scalar multiplication
   $\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B)$

(3) Kronecker product and matrix multiplication
$(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$

Comment I: From the right to the left the condition on the validity of the left is needed.

Ex2: Let $A = B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $C = D = (0, 1)$. Then $(A \otimes C)(B \otimes D) = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}^2$.
But in $(AB) \otimes (CD)$ both AB and CD are not valid expressions.

Comment II: Multiple factors in matrix product
$(A_1 A_2 \cdots A_k) \otimes (B_1 B_2 \cdots B_k) = (A_1 \otimes B_1)(A_2 \otimes B_2) \cdots (A_k \otimes B_k)$

Comment III: Multiple factors in kronecker products
$(A_1 A_2) \otimes (B_1 B_2) \otimes \cdots \otimes (C_1 C_2) = (A_1 \otimes B_1 \otimes \cdots \otimes C_1)(A_2 \otimes B_2 \otimes \cdots \otimes C_2)$.

Comment IV: Multiple factors in both matrix multiplication and in Kronecker product
$(A_{11} \cdots A_{1k}) \otimes (A_{21} \cdots A_{2k}) \otimes \cdots \otimes (A_{n1} \cdots A_{nk})$
$= (A_{11} \otimes \cdots \otimes A_{n1})(A_{12} \otimes \cdots \otimes A_{n2}) \cdots (A_{1k} \otimes \cdots \otimes A_{n_k})$

3. Kronecker product of one row vector and one column vector
Let x, y and z be column vectors.

(1) While generally $A \otimes B \neq B \otimes A$, for column vectors x and y,

$$x \otimes y' = y' \otimes x = xy' \text{ and } x \otimes y^* = y^* \otimes x = xy^*$$

Pf: Show the second equation system.
For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $x \otimes y^* = \begin{pmatrix} x_1 y^* \\ \vdots \\ x_n y^* \end{pmatrix} = \begin{pmatrix} x_1 \overline{y}_1 & \cdots & x_1 \overline{y}_n \\ \vdots & \ddots & \vdots \\ x_n \overline{y}_1 & \cdots & x_n \overline{y} \end{pmatrix} = xy^*$
and $y^* \otimes x = \begin{pmatrix} \overline{y}_1 x & \cdots & \overline{y}_n x \end{pmatrix} = \begin{pmatrix} x_1 \overline{y}_1 & \cdots & x_1 \overline{y}_n \\ \vdots & \ddots & \vdots \\ x_n \overline{y}_1 & \cdots & x_n \overline{y} \end{pmatrix} = xy^*$

(2) $\mathrm{vec}(x \otimes y') = \mathrm{vec}(y' \otimes x) = \mathrm{vec}(xy') = y \otimes x$
$\mathrm{vec}(x \otimes y^*) = \mathrm{vec}(y^* \otimes x) = \mathrm{vec}(xy^*) = \overline{y} \otimes x$

Pf: Show the first one
$\mathrm{vec}(xy') = \mathrm{vec}\left[\begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix}\right] = \begin{pmatrix} y_1 x \\ \vdots \\ y_n x \end{pmatrix} = y \otimes x.$

(3) $x' \otimes y \otimes z = y \otimes x' \otimes z = y \otimes z \otimes x' = (y \otimes z)x'$
$x^* \otimes y \otimes z = y \otimes x^* \otimes z = y \otimes z \otimes x^* = (y \otimes z)x^*$
$\mathrm{vec}(x' \otimes y \otimes z) = \mathrm{vec}(y \otimes z \otimes x') = \mathrm{vec}[(y \otimes z)x'] = x \otimes y \otimes z$
$\mathrm{vec}(x^* \otimes y \otimes z) = \mathrm{vec}(y \otimes z \otimes x^*) = \mathrm{vec}[(y \otimes z)x^*] = \overline{x} \otimes y \otimes z$

Ex3: For $x \in C^n$ and $y \in C^n$ find $(x \otimes y')^2$ and $\mathrm{vec}[(x \otimes y')^2]$.

$$(x \otimes y')^2 = (x \otimes y')(x \otimes y') = (y' \otimes x)(x \otimes y') = (y'x) \otimes (xy') = y'xxy'$$

$$\mathrm{vec}[(x \otimes y')^2] = \mathrm{vec}(y'xxy') = (y'x)(y \otimes x).$$

L23 Eigenvalues, ranks, trace and determinants of $A \otimes B$

1. More on $A \otimes B$

   (1) $\overline{A \otimes B} = \overline{A} \otimes \overline{B}$. Recall that $\overline{AB} = \overline{A}\,\overline{B}$.

   Proof With $A = (a_{ij})_{m\times n}$, $\overline{A \otimes B}$ has $(i,\ j)$ block $\overline{a_{ij}B}$. $\overline{A} \otimes \overline{B}$ has $(i,\ j)$ block $\overline{a}_{ij}\,\overline{B}$.
   But $\overline{a_{ij}B} = \overline{a}_{ij}\,\overline{B}$. So $\overline{A \otimes B} = \overline{A} \otimes \overline{B}$.

   (2) $(A \otimes B)' = A' \otimes B'$. Recall that $(AB)' = B'A'$.

   Proof $(A \otimes B)' = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}' = \begin{pmatrix} a_{11}B' & \cdots & a_{m1}B' \\ \vdots & \ddots & \vdots \\ a_{1n}B' & \cdots & a_{mn}B' \end{pmatrix} = A' \otimes B'$.

   Ex1: If both A and B are symmetric, so is $A \otimes B$ since $(A \otimes B)' = A' \otimes B' = A \otimes B$.

   (3) $(A \otimes B)^* = A^* \otimes B^*$. Recall $(AB)^* = B^*A^*$.

   Proof $(A \otimes B)^* = \left[\overline{(A \otimes B)}\right]' = \left(\overline{A} \otimes \overline{B}\right)' = \left(\overline{A}\right)' \otimes \left(\overline{B}\right)' = A^* \otimes B^*$

   Ex2: If both A and B are Hermitian so is $A \otimes B$ since $(A \otimes B)^* = A^* \otimes B^* = A \otimes B$.

   (4) $(A \otimes B)^+ = A^+ \otimes B^+$. Under special conditions $(AB)^+ = B^+A^+$.

   Proof For $A \otimes B$ let $G = A^+ \otimes B^+$. One can show that the four Penrose conditions are satisfied. Here we show (iii) $(A \otimes B)G$ is Hermitian.
   $(A \otimes B)G = (A \otimes B)(A^+ \otimes B^+) = (AA^+) \otimes (BB^+)$ is Hermitian since both AA^+ and BB^+ are Hermitian.

2. Schur-decomposition of $A \otimes B$

   (1) If A and B are both unitary matrices, so is $A \otimes B$.

   Proof If A and B are both unitary, then $A^* = A^{-1}$ and $B^* = B^{-1}$.
   So $(A \otimes B)^*(A \otimes B) = (A^*A) \otimes (B^*B) = I_m \otimes I_n = I_{mn}$.
   Thus $(A \otimes B)^* = (A \otimes B)^{-1}$. Hence $A \otimes B$ is unitary.

   (2) If $A \in C^{m\times m}$ and $B \in C^{n\times n}$ are both upper-triangular, so is $A \otimes B$.

   Proof If A and B are both upper-triangular, then
   $A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}B \end{pmatrix}$ is also upper-trianglular.

   Comments: If both A and B are lower-triangular, so is $A \otimes B$.
   If both A and B are diagonal, so is $A \otimes B$.

   (3) If $A = XDX^*$ and $B = YTY^*$ are Schur-decompositions for A and B,
   then $A \otimes B = (X \otimes Y)(D \otimes T)(X \otimes Y)^*$ is the Schur-decomposition for $A \otimes B$.

   Pf: $A \otimes B = (XDX^*) \otimes (YTY^*) = (X \otimes Y)(D \otimes T)(X \otimes Y)^*$.
   X and Y are unitary$\Longrightarrow X \otimes Y$ is unitary.
   D and T are upper-triangular$\Longrightarrow D \otimes T$ is upper-trianglular.
   So $A \otimes B = (X \otimes Y)(D \otimes T)(X \otimes Y)^*$ is a Schur-decomposition.

   Comment: $A \otimes B$ and $D \otimes T$ are similar.

3. Eigenvalues, ranks, traces and determinants of $A \otimes B$

(1) Eigenvalues of $A \otimes B$ and $B \otimes A$

If $\lambda_1, ..., \lambda_m$ are eigenvalues for $A \in C^{m\times m}$ and $\gamma_1, ..., \gamma_n$ are eigenvalues for $B \in C^{n\times n}$, then $\lambda_i\gamma_j$, $i = 1, .., m$ and $j = 1, ..., n$ are eigenvalues for $A \otimes B$.

Pf: $\lambda_1, ..., \lambda_m$ and $\gamma_1, ..., \gamma_n$ are eigenvalues for A and B with Schur-decompositions $A = XDX^*$ and $B = YTY^*$. Then $\lambda_1, ..., \lambda_m$ and $\gamma_1, ..., \gamma_n$ are diagonal elements of D and T. So $\lambda_i\gamma_j$, $i = 1, .., m$ and $j = 1, ..., n$ are diagonal elements of $D \otimes T$ in Shur-decomposition $A \otimes B = (X \otimes Y)(D \otimes T)(X \otimes Y)^*$. So $\lambda_i\gamma_j$, $i = 1, .., m$ and $j = 1, .., n$ are eigenvalues of $A \otimes B$. Similarly they are eigenvalues of $B \otimes A$.

$$\begin{array}{c|ccc} & \gamma_1 & \cdots & \gamma_n \\ \hline \lambda_1 & \lambda_1\gamma_1 & \cdots & \lambda_1\gamma_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m & \lambda_m\gamma_1 & \cdots & \lambda_m\gamma_n \end{array}$$

(2) $\text{rank}(A \otimes B) = \text{rank}(A)\,\text{rank}(B) = \text{rank}(B \otimes A)$

If $A \in C^{m\times n}$ and $B \in C^{p\times q}$, then $\text{rank}(A \otimes B) = \text{rank}(A)\,\text{rank}(B) = \text{rank}(B \otimes A)$.

Pf: $\text{rank}(A) = \text{rank}(AA^+) = \#$ of non-zero in $\lambda_1, .., \lambda_m$, the eigenvalues of AA^+.
$\text{rank}(B) = \text{rank}(BB^+) = \#$ of non-zero in $\gamma_1, ..., \gamma_p$, the eigenvalues of BB^+.
$\text{rank}(A \otimes B) = \text{rank}[(A \otimes B)(A \otimes B)^+] = \text{rank}[(AA^+) \otimes (BB^+)]$
$= \#$ of non-zero in $\lambda_i\gamma_j$, $i = 1, .., m$; $j = 1, .., p$, the eigenvalues of $(AA^+) \otimes (BB^{)}$.
But this number is the product of the previous two.

(3) $\text{tr}(A \otimes B) = \text{tr}(A)\,\text{tr}(B) = \text{tr}(B \otimes A)$

If $A \in C^{m\times m}$ and $B \in C^{n\times n}$, then $\text{tr}(A \otimes B) = \text{tr}(A)\,\text{tr}(B) = \text{tr}(B \otimes A)$.

Pf: Suppose A has eigenvalues $\lambda_1, .., \lambda_m$ and B has eigenvalues $\gamma_1, ..., \gamma_n$. Then
$$\begin{aligned} \text{tr}(A \otimes B) &= [(\lambda_1\gamma_1) + \cdots + (\lambda_1\gamma_n)] + \cdots + [(\lambda_m\gamma_1) + \cdots + (\lambda_m\gamma_n)] \\ &= \lambda_1\text{tr}(B) + \cdots + \lambda_m\text{tr}(B) = \text{tr}(A)\,\text{tr}(B) = \text{tr}(B \otimes A) \end{aligned}$$

(4) $|A \otimes B| = |A|^n\,|B|^m = |B \otimes A|$

If $A \in C^{m\times m}$ and $B \in C^{n\times n}$, then $|A \otimes B| = |A|^n\,|B|^m$

Pf: With the assumption on the eigenvalues in the proof of (3)
$$\begin{aligned} |A \otimes B| &= [(\lambda_1\gamma_1)\cdots(\lambda_1\gamma_n)]\cdots[(\lambda_m\gamma_1)\cdots(\lambda_m\gamma_n)] \\ &= \lambda_1^n|B|\cdots\lambda_m^n|B| = |A|^n\,|B|^m \end{aligned}$$

Comments: If $A \in C^{m\times n}$ and $B \in C^{n\times m}$ with $m \neq n$, then A and B do not have eigenvalues, traces and determinants. But $A \otimes B \in C^{mn\times mn}$ and $B \otimes A \in C^{mn\times mn}$ do.

(1) $A \otimes B$ and $B \otimes A$ may no longer share the eigenvalues.

(ii) $\text{rank}(A \otimes B) = \text{rank}(A)\,\text{rank}(B) = \text{rank}(B \otimes A)$ holds for $A \in C^{m\times n}$ and $B \in C^{p\times q}$.

(iii) When $A \in C^{m\times n}$ and $B \in C^{n\times m}$, one can find an example of $\text{tr}(A \otimes B) \neq \text{tr}(B \otimes A)$. This is a HW problem.

(iv) When $A \in C^{m\times n}$ and $B \in C^{n\times m}$, $|A \otimes B| = |B \otimes A|$ holds. This is also a HW problem.