**L19 Orthogonal complements of matrix spaces**

1. Cross expressions

   (1) Conditions for expressing ranges as null spaces
   If $D$ is idempotent, i.e., $D^2 = D$, then

   $$\mathcal{R}(D,\, I_n) = \mathcal{N}(I - D,\, I_n) \text{ and } \mathcal{R}(I_m,\, D) = \mathcal{N}(I_m,\, I - D)$$

   **Pf:** Show the second equation.
   $\subset$: If $Y \in \mathcal{R}(I_m,\, D)$, then $Y = I_m XD$. So $I_m Y(I - D) = XD - XD^2 = 0$.
   Thus $Y \in \mathcal{N}(I_m,\, I - D)$.
   $\supset$: If $Y \in \mathcal{N}(I_m,\, I - D)$, then $Y(I - D) = 0$. Thus $Y = YD \in \mathcal{R}(I_m,\, D)$.

   (2) Conditions for expressing null spaces as ranges
   If $D$ is idempotent, then

   $$\mathcal{N}(D,\, I_q) = \mathcal{R}(I - D,\, I_q) \text{ and } \mathcal{N}(I_p,\, D) = \mathcal{R}(I_p,\, I - D)$$

   **Pf:** Show the second equation.
   $D$ is idempotent $\Longrightarrow$ $I - D$ is idempotent. So by the second equation in (1) 1, $\mathcal{R}(I_p,\, I - D) = \mathcal{N}(I_p,\, I - (I - D)) = \mathcal{N}(I_p,\, D)$. So $\mathcal{N}(I_p,\, D) = \mathcal{R}(I_p,\, I - D)$.

   (3) Expressing the ranges as null spaces
   $\mathcal{R}(A,\, I_n) = \mathcal{R}(AA^-,\, I_n) = \mathcal{N}(I_m - AA^-,\, I_n)$.
   $\mathcal{R}(I_m,\, B) = \mathcal{R}(I_m,\, B^-B) = \mathcal{N}(I_m,\, I_n - B^-B)$.
   $\mathcal{R}(A,\, B) = \mathcal{R}(A,\, I_n) \cap \mathcal{R}(I_m,\, B) = \mathcal{N}(I - AA^-,\, I_n) \cap \mathcal{N}(I_m,\, I - B^-B)$.

   (4) Expressing the null spaces as ranges
   $\mathcal{N}(A,\, I_q) = \mathcal{N}(A^-A,\, I_q) = \mathcal{R}(I - A^-A,\, I_q)$
   $\mathcal{N}(I_p,\, B) = \mathcal{N}(I_p,\, BB^-) = \mathcal{R}(I_p,\, I_q - BB^-)$
   $\mathcal{N}(A,\, B) = \mathcal{R}(I - A^-A,\, I) + \mathcal{R}(I,\, I - BB^-)$

   **Ex1:** $\mathcal{R}(A,\, I_n) = \mathcal{R}(AA^+,\, I_n) = \mathcal{N}(I_m - AA^+,\, I_n)$
   $\mathcal{R}(I_m,\, B) = \mathcal{R}(I_m,\, B^+B) = \mathcal{N}(I_m,\, I_n - B^+B)$
   $\mathcal{R}(A,\, B) = \mathcal{R}(A,\, I_n) \cap \mathcal{R}(I_m,\, B) = \mathcal{N}(I - AA^+,\, I_n) \cap \mathcal{N}(I_m,\, I - B^+B)$
   $\mathcal{N}(A,\, I_q) = \mathcal{N}(A^+A,\, I_q) = \mathcal{R}(I - A^+A,\, I_q)$
   $\mathcal{N}(I_p,\, B) = \mathcal{N}(I_p,\, BB^+) = \mathcal{R}(I_p,\, I_q - BB^+)$
   $\mathcal{N}(A,\, B) = \mathcal{R}(I - A^+A,\, I) + \mathcal{R}(I,\, I - BB^+)$

   **Ex2:** Solutions to $AXB = 0$.

   (i) Two-parameter version

   $$\begin{aligned} AXB = 0 \quad &\Longleftrightarrow \quad X \in \mathcal{N}(A,\, B) = \mathcal{R}(I_p - A^+A,\, I_q) + \mathcal{R}(I_p,\, I_q - BB^+) \\ &\Longleftrightarrow \quad X \in \{(I_p - A^+A)H_1 + H_2(I - BB^+) :\, H_1,\, H_2 \in C^{p\times q}\}. \end{aligned}$$

   (ii) One-parameter version:
   By the result in HW,

   $$AXB = 0 \Longleftrightarrow X \in \mathcal{N}(A,\, B) = \{H - A^+AHBB^+ :\, H \in C^{p\times q}\}.$$

2. Orthogonal complement

   (1) Frobenius inner product for matrices
   For $X = (x_{ij})_{m\times n}$ and $Y = (y_{ij})_{m\times n}$, $\langle X,\, Y\rangle = \sum_i \sum_j x_{ij}\overline{y}_{ij} = \mathrm{tr}(Y^*X)$ satisfies

      (i) $\langle X,\, Y\rangle = \overline{\langle Y,\, X\rangle}$

      (ii) $\langle \alpha X + \beta Y,\, Z\rangle = \alpha\langle X,\, Z\rangle + \beta\langle Y,\, Z\rangle$

      (iii) $\langle X,\, X\rangle \geq 0$; $\langle X,\, X\rangle = 0 \Longleftrightarrow X = 0$

   and hence is an inner product called the Frobenius inner product.

   (2) Induced norm and angle
   With the inner product, the norm $\|X\| = \sqrt{\langle X,\, X\rangle}$ the angle formed by $X$ and $Y$, $\theta \in (-\pi,\, \pi]$ satisfying $\cos(\theta) = \frac{\langle X,\, Y\rangle}{\|X\|\,\|Y\|}$ are defined.

   (3) Orthogonal complements
   Suppose $\mathcal{S}$ is a subspace.
   $X \perp Y \overset{def}{\Longleftrightarrow} \langle X,\, Y\rangle = 0$
   $X \perp \mathcal{S} \overset{def}{\Longleftrightarrow} X \perp Y$ for all $Y \in \mathcal{S}$. $\mathcal{S}^\perp \overset{def}{=} \{X : X \perp \mathcal{S}\} = \{X : X \perp Y \text{ for all } Y \in \mathcal{S}\}$
   is called the orthogonal complement of $\mathcal{S}$.

3. Orthogonal complements of ranges and null spaces

   (1) $\mathcal{R}^\perp(A,\, B) = \mathcal{N}(A^*,\, B^*)$.

   **Pf:** 
   $$\begin{aligned} Z \in \mathcal{R}^\perp(A,\, B) \quad &\overset{def}{\Longrightarrow} \quad Z \perp \mathcal{R}(A,\, B) \Longleftrightarrow Z \perp AXB \text{ for all } X \in C^{p\times q} \\ &\Longleftrightarrow \quad 0 = \langle Z,\, AXB\rangle = \mathrm{tr}(B^*X^*A^*Z) \text{ for all } X \in C^{p\times q} \\ &\Longleftrightarrow \quad 0 = \mathrm{tr}(X^*A^*ZB^*) = \langle A^*ZB^*,\, X\rangle \text{ for all } X \in C^{p\times q} \\ &\overset{*}{\Longleftrightarrow} \quad A^*ZB^* = 0 \Longleftrightarrow Z \in \mathcal{N}(A^*,\, B^*). \end{aligned}$$
   $\overset{*}{\Longrightarrow}$: Take $X = A^*ZB^*$. $\quad \overset{*}{\Longleftarrow}$: Trivial.

   (2) $\mathcal{N}^\perp(A,\, B) = \mathcal{R}(A^*,\, B^*)$

   **Pf:** In (1) replace $A$ and $B$ by $A^*$ and $B^*$. $\mathcal{R}^\perp(A^*,\, B^*) = \mathcal{N}(A,\, B)$.
   Take the orthogonal complements of both sides. $\mathcal{R}(A^*,\, B^*) = \mathcal{N}^\perp(A,\, B)$.

**Ex3:** $\mathcal{R}^\perp(A,\, I_n) = \mathcal{N}(A^*,\, I_n) = \mathcal{N}(AA^+,\, I_n)$. $\mathcal{R}^\perp(A,\, I_n) = \mathcal{R}^\perp(AA^+,\, I_n) = \mathcal{N}(AA^+,\, I_n)$
$\mathcal{R}^\perp(I_m,\, B) = \mathcal{N}(I_m,\, B^*) = \mathcal{N}(I_m,\, B^+B)$. $\mathcal{R}^\perp(I_m,\, B) = \mathcal{R}^\perp(I_m,\, B^+B) = \mathcal{N}(I_m,\, B^+B)$.
$\mathcal{R}^\perp(A,\, B) = \mathcal{N}(A^*,\, B^*) = \mathcal{N}(AA^+,\, B^+B)$.
$\mathcal{R}^\perp(A,\, B) = \mathcal{R}^\perp(AA^+,\, B^+B) = \mathcal{N}(AA^+,\, B^+B)$.

$\mathcal{N}^\perp(A,\, I_q) = \mathcal{R}(A^*,\, I_q) = \mathcal{R}(A^+A,\, I_q)$. $\mathcal{N}^\perp(A,\, I_q) = \mathcal{N}^\perp(A^+A,\, I_q) = \mathcal{R}(A^+A,\, I_q)$
$\mathcal{N}^\perp(I_p,\, B) = \mathcal{R}(I_p,\, B^*) = \mathcal{R}(I_p,\, BB^+)$ $\mathcal{N}^\perp(I_p,\, B) = \mathcal{N}^\perp(I_p,\, BB^+) = \mathcal{R}(I_p,\, BB^|)$.
$\mathcal{N}^\perp(A,\, B) = \mathcal{R}(A^*,\, B^*) = \mathcal{R}(A^+A,\, BB^+)$.
$\mathcal{N}^\perp(A,\, B) = \mathcal{N}^\perp(A^+A,\, BB^+) = \mathcal{R}(A^+A,\, BB^+)$.

**L20 Matrix projections**

1. Projections

   (1) More on orthogonal complement
   Let $S_1$, $S_2$ and $S$ be subspaces of finite dimensional $V$.
      (i) $(S^\perp)^\perp = S$.
      (ii) $S_1 \subset S_2 \Longrightarrow S_2^\perp \subset S_1^\perp$.
      **Proof** $Z \in S_2^\perp \Longrightarrow Z \perp S_2 \Longrightarrow Z \perp S_1 \Longrightarrow Z \in S_1^\perp$.
      (iii) $(S_1 + S_2)^\perp = S_1^\perp \cap S_2^\perp$
      **Proof** $\subset$: $S_i \subset S_1 + S_2 \Longrightarrow (S_1 + S_2)^\perp \subset S_i^\perp, \Longrightarrow (S_1 + S_2)^\perp \subset S_1^\perp \cap S_2^\perp$.
      $\supset$: $Y \in S_1^\perp \cap S_2^\perp \Longrightarrow Y \perp S_i \Longrightarrow Y \perp S_1 + S_2 \Longrightarrow Y \in (S_1 + S_2)^\perp$.
      (iv) $(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$.
      **Proof** In (iii) replacing $S_i$ by $S_i^\perp$ and taking orthogonal complements of both sizes lead to the result.

   (2) Projections
   $V$ is a space in which inner product and its induced norm are defined. $S$ is a subspace of $V$. For $Y \in V$ there exists a unique $\widehat{Y} \in S$ such that
   $$\|Y - \widehat{Y}\|^2 \leq \|Y - Z\|^2 \text{ for all } Z \in S.$$
   This $\widehat{Y}$ is called the projection of $Y$ onto $S$ and is denoted as $\pi(Y|S)$.

   (3) A sufficient condition
   For $Y \in V$ if $Y = \widehat{Y} + \widetilde{Y}$ where $\widehat{Y} \in \mathcal{S}$ and $\widetilde{Y} \in \mathcal{S}^\perp$, then $\widehat{Y} = \pi(Y|\mathcal{S})$ and $\widetilde{Y} = \pi(Y|\mathcal{S}^\perp)$.
   **Proof** $\widehat{Y} \in \mathcal{S}$ and $Y - \widehat{Y} \in \mathcal{S}^\perp$. By (3) of 3 in L16, $\widehat{Y} = \pi(Y \mid \mathcal{S})$.
   $\widetilde{Y} \in S^\perp$ and $Y - \widetilde{Y} \in \mathcal{S} = [\mathcal{S}^\perp]^\perp$. By (3) of 3 in L16, $\widetilde{Y} = \pi(Y \mid S^\perp)$.

2. Projections onto $\mathcal{R}(A, B)$ and $\mathcal{N}(A, B)$

   (1) $\pi(Y|\mathcal{R}(A, B)) = AA^+YB^+B$ and $\pi(Y|\mathcal{R}^\perp(A, B)) = Y - AA^+YB^+B$.
   **Proof** $Y = AA^+YB^+B + (Y - AA^+YB^+B)$ where $AA^+YB^+B \in \mathcal{R}(A, B)$.
   $A^*(Y - AA^+YB^+B)B^* = A^*YB^* - A^*YB^* = 0$.
   So $Y - AA^+YB^+B \in \mathcal{N}(A^*, B^*) = \mathcal{R}^\perp(A, B)$. Conclusion follows.

   (2) $\pi(X|\mathcal{N}(A, B)) = X - A^+AXBB^+$ and $\pi(X|\mathcal{N}^\perp(A, B)) = A^+AXBB^+$.
   **Proof** $X = (X - A^+AXBB^+) + A^+AXBB^+$ where $X - A^+AXBB^+ \in \mathcal{N}(A, B)$.
   But $A^+AXBB^+ \in \mathcal{R}(A^+A, BB^+) = \mathcal{R}(A^*, B^*) = \mathcal{N}^\perp(A, B)$.
   Conclusion follows.

   **Comments** Two formulas
   $$\pi(Y \mod \mathcal{R}(A, B)) = AA^+YB^+B \text{ and } \pi(X \mid \mathcal{N}(A, B)) = X - A^+AXBB^+.$$
   The other two are by the relations $\pi(Y \mid \mathcal{R}^\perp(A, B)) = Y - \pi(Y \mod \mathcal{R}(A, B))$ and $\pi(X \mod \mathcal{N}^\perp(A, )) = X - \pi(X \mod \mathcal{N}(A, B))$

   **Ex1:** $\pi(Y|\mathcal{R}(A, I_n)) = AA^+YI^+I = AA^+Y$
   $\pi(Y|\mathcal{R}(I_m, B)) = II^+YB^+B = YB^+B$
   $\pi(Y|\mathcal{R}^\perp(A, I_n)) = Y - AA^+Y = (I - AA^+)Y$.
   $\pi(Y|\mathcal{R}^\perp(I_m, B)) = Y - YB^+B = Y(I_n - B^+B)$.

**Ex2:** $\pi(y|\mathcal{R}(A)) = \pi(y|\mathcal{R}(A,\,1)) = AA^+y$ and $\pi(y|\mathcal{R}^\perp(A)) = (I - AA^+)y$.

**Ex3:** $\pi(X|\mathcal{N}(A,\,I_q)) = X - A^+AXII^+ = (I_p - A^+A)X$
$\pi(X|\mathcal{N}(I_p,\,B)) = X - I^+IXBB^+ = X(I_q - BB^+)$
$\pi(X|\mathcal{N}^\perp(A)) = A^+AX$.
$\pi(X|\mathcal{N}^\perp(I_p,\,B)) = XBB^+$.

**Ex4:** $\pi(x|\mathcal{N}(A)) = \pi(x|\mathcal{N}(A,\,1)) = (I - A^+A)x$
and $\pi(x|\mathcal{N}^\perp(A)) = \pi(x|\mathcal{N}^\perp(A,\,1)) = A^+Ax$.

3. Projections onto $\mathcal{R}(A,\,I_n) + \mathcal{R}(I_m,\,B)$ and $\mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B)$

(1) $\pi(Y|\mathcal{R}(A,\,I_n) + \mathcal{R}(I_m,\,B)) = AA^+Y + YB^+B - AA^+YB^+B$

**Proof:** Note that

$$\begin{aligned}
\mathcal{R}(A,\,I_n) + \mathcal{R}(I_m,\,B) &= \mathcal{R}(AA^+,\,I_n) + \mathcal{R}(I_m,\,B^+B) \\
&= \mathcal{N}(I_m - AA^+,\,I_n) + \mathcal{N}(I_m,\,I_n - B^+B) \\
&= \mathcal{N}(I - AA^+,\,I - B^+B)
\end{aligned}$$

$$\begin{aligned}
\text{So } \pi(Y \mid \mathcal{R}(A,\,I) + \mathcal{R}(I,\,B)) &= \pi(Y \mid \mathcal{N}(I - AA^+,\,I - B^+B)) \\
&= Y - (I - AA^+)Y(I - B^+B) \qquad . \\
&= AA^+Y + YB^+B - AA^+YB^+B
\end{aligned}$$

(2) $\pi(X|\mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B)) = (I_p - A^+A)X(I_q - BB^+)$
and $\pi(X|[\mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B)]^\perp) = A^+AX + XBB^+ - A^+AXBB^+$.

**Proof:** Note that

$$\begin{aligned}
\mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B) &= \mathcal{N}(A^+A,\,I_q) \cap \mathcal{N}(I_p,\,BB^+) \\
&= \mathcal{R}(I_p - A^+A,\,I_q) \cap \mathcal{R}(I_p,\,I_q - BB^+) \\
&= \mathcal{R}(I_p - A^+A,\,I_q - BB^+).
\end{aligned}$$

$$\begin{aligned}
\text{So } \pi(X|\mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B)) &= \pi(X|\mathcal{R}(I_p - A^+A,\,I_q - BB^+)) \\
&= (I_p - A^+A)X(I_q - BB^+).
\end{aligned}$$

**Comments:** Two formulas

$$\begin{aligned}
\pi(Y| \mid \mathcal{R}(A,\,I) + \mathcal{R}(I,\,B)) &= AA^+Y + YBB^+ - AA^+YB^+B \\
&= \pi(Y \mid \mathcal{R}(A,\,I)) + \pi(Y \mid \mathcal{R}(I,\,B)) - \pi(Y \mid \mathcal{R}(A,\,B))
\end{aligned}$$

and $\pi(X \mid \mathcal{N}(A,\,I) \cap \mathcal{N}(I,\,B)) = (I - A^+A)X(I - BB^+)$.
The other two are by the relation
$\pi(Y \mid [\mathcal{R}(A,\,I) + \mathcal{R}(I,\,B)]^\perp) = Y - \pi(Y \mid \mathcal{R}(A,\,I) + \mathcal{R}(I,\,B))$.
$\pi(X| \,[\mathcal{N}(A,\,I) \cap \mathcal{N}(I,\,B)]^\perp) = X - \pi(X \mid \mathcal{N}(A,\,I) \cap \mathcal{N}(I,\,B))$

**Ex5:** Find $\pi(X|\mathcal{R}^\perp(A^*,\,I_q) \cap \mathcal{N}(I_p,\,(B^*)^+))$.
$\mathcal{R}^\perp(A^*,\,I_q) \cap \mathcal{N}(I_p,\,(B^*)^+) = \mathcal{N}(A,\,I_q) \cap \mathcal{N}(I_p,\,B)$.
So, $\pi(X|\mathcal{R}^\perp(A^*,\,I_q) \cap \mathcal{N}(I_p,\,(B^*)^+)) = (I_p - A^+A)X(I_q - BB^+)$.