L15: $\mathcal{R}(A)$, $\mathcal{N}(A)$ and their relations

1. $\mathcal{R}(A)$

   (1) A subspace of C^m: $\mathcal{R}(A)$
   $y = Ax$ with $A \in C^{m\times n}$ is a linear transformation (LT) from $x \in C^n$ to $y \in C^m$. The range of this LT
   $$\mathcal{R}(A) = \{Ax : x \in C^n\}$$
   is closed under LCs and hence is a LS in C^m with $\dim[\mathcal{R}(A)] = \text{rank}(A)$. $\mathcal{R}(A)$ is the span of the columns of A also denoted as $\mathcal{C}(A)$, $\mathcal{S}(A)$ or $L(A)$ for a LS from A.

   Ex1: In linear regression $y = X\beta + \epsilon$ with $E(\epsilon) = 0$, $E(y) = X\beta$. We can say now, $E(y)$ is a LT of unknown β and lies in a known space $\mathcal{R}(A)$ with dimension rank(A).

   (2) $\mathcal{R}(A) = \mathcal{R}(AA^-)$

   $\subset$**:** $y \in \mathcal{R}(A) \Longrightarrow y = Ax = AA^-(Ax)$ for some $x \Longrightarrow y \in \mathcal{R}(AA^-)$.

   $\supset$**:** $y \in \mathcal{R}(AA^-) \Longrightarrow y = AA^-x = A(A^-x)$ for some $x \Longrightarrow y \in \mathcal{R}(A)$.

   Comment: $\mathcal{R}(A) = \mathcal{R}(AA^-) = \mathcal{R}(AA^+)$. Properties of AA^- and AA^+.

   (3) $\mathcal{R}(A) = \mathcal{R}(AA^*)$

   Pf: Note that $\mathcal{R}(AB) \subset \mathcal{R}(A)$ since $y \in \mathcal{R}(AB) \Longrightarrow y = ABx = A(Bx) \in \mathcal{R}(A)$. So $\mathcal{R}(A) = \mathcal{R}(AA^+A) = \mathcal{R}(AA^*(A^+)^*) \subset \mathcal{R}(AA^*) \subset \mathcal{R}(A)$.

   (4) $\mathcal{R}(A) = \mathcal{R}((A^+)^*)$
   This can be proved by either the method in (2) or in (3).

   (5) Summary
   $\mathcal{R}(A) = \{y = Ax : x\} = \mathcal{C}(A) = \mathcal{S}(A) = L(A)$ is a subspace in C^m. The member of this space y is determined by its structure $y = Ax$.
   $\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^-) = \mathcal{R}\left((A^*)^+\right)$.

   Ex2: $\mathcal{R}(A^*) = \mathcal{R}((A^{**})^+) = \mathcal{R}(A^+)$. $\quad$ $\mathcal{R}(A^+) = \mathcal{R}(A^+(A^+)^+) = \mathcal{R}(A^+A)$.
   So $\mathcal{R}(A^*) = \mathcal{R}(A^+) = \mathcal{R}(A^+A)$.

2. $\mathcal{N}(A)$

   (1) A subspace of C^n: $\mathcal{N}(A)$
   For LT $y = f(x) = Ax$, the Kernel, Kernel$(f) = \{x : f(x) = 0\}$, is also called the null of space of A,
   $$\mathcal{N}(A) = \{x \in C^n : Ax = 0\}.$$
   $\mathcal{N}(A)$ is a subspace of C^n with $\dim[\mathcal{N}(A)] = n - \text{rank}(A)$.

   Proof If u, $v \in \mathcal{N}(A)$, then $Au = 0$ and $Av = 0$. Hence $A(\alpha u + \beta v) = \alpha Au + \beta Av = 0$. thus $\alpha u + \beta v \in \mathcal{N}(A)$.

   Ex3: The collection of all solutions to the equation $Ax = 0$ form a linear space in C^n, $\mathcal{N}(A)$.

   (2) $\mathcal{N}(A) = \mathcal{N}(A^-A)$.

   $\subset$**:** $x \in \mathcal{N}(A) \Longrightarrow Ax = 0 \Longrightarrow A^-Ax = 0 \Longrightarrow x \in \mathcal{N}(A^-A)$.

   $\supset$**:** $x \in \mathcal{N}(A^-A) \Longrightarrow A^-Ax = 0 \Longrightarrow Ax = AA^-Ax = 0 \Longrightarrow x \in \mathcal{N}(A)$.

   Comment: $\mathcal{N}(A) = \mathcal{N}(A^-A) = \mathcal{N}(A^+A)$. Properties of A^-A and A^+A.

(3) $\mathcal{N}(A) = \mathcal{N}(A^*A)$

Proof Note that $\mathcal{N}(A) \subset \mathcal{N}(BA)$ since $Ax = 0 \Longrightarrow BAx = 0$.
So $\mathcal{N}(A) \subset \mathcal{N}(A^*A) \subset \mathcal{N}((A^+)^*A^*A) = \mathcal{N}(A)$.

(4) $\mathcal{N}(A) = \mathcal{N}((A^+)^*)$
This can be proved by either the method in (2) or the method in (3).

(5) Summary
$\mathcal{N}(A) = \{x : Ax = 0\}$ is a subspace of C^n. Its member x is determined by its property $Ax = 0$.
$\mathcal{N}(A) = \mathcal{N}(A^*A) = \mathcal{N}(A^-A) = \mathcal{N}(A^+A) = \mathcal{N}\left((A^+)^*\right)$.

Ex4: $\mathcal{N}(A^*) = \mathcal{N}((A^{**})^+) = \mathcal{N}(A^+)$ $\quad$ $\mathcal{N}(A^+) = \mathcal{N}((A^+)^+A^+) = \mathcal{N}(AA^+)$.

3. Cross expressions

(1) Lemma I
If D is idempotent, i.e., $D^2 = D$, then $\mathcal{R}(D) = \mathcal{N}(I - D)$.

Proof: $\subset$**:** $y \in \mathcal{R}(D) \Longrightarrow y = Dx \Longrightarrow (I - D)y = Dx - D^2x = 0 \Longrightarrow y \in \mathcal{N}(I - D)$.
$\supset$**:** $x \in \mathcal{N}(I - D) \Longrightarrow (I - D)x = 0 \Longrightarrow x = Dx \in \mathcal{R}(D)$.

(2) Lemma II
If $D^2 = D$, then $\mathcal{N}(D) = \mathcal{R}(I - D)$.

Proof: If $D^2 = D$, then $(I - D)^2 = (I - D)(I - D) = I - D - D + D^2 = I - D$.
By (1), $\mathcal{R}(I - D) = \mathcal{N}\left(I - (I - D)\right) = \mathcal{N}(D)$. Thus $\mathcal{N}(D) = \mathcal{R}(I - D)$.

(3) $\mathcal{R}(A) = \mathcal{N}(\cdot)$
$\mathcal{R}(A) = \mathcal{R}(AA^-) = \mathcal{N}(I_m - AA^-)$ $\quad$ $\mathcal{R}(A) = \mathcal{R}(AA^+) = \mathcal{N}(I_m - AA^+)$.

(4) $\mathcal{N}(A) = \mathcal{R}(\cdot)$
$\mathcal{N}(A) = \mathcal{N}(A^-A) = \mathcal{R}(I_n - A^-A)$ $\quad$ $\mathcal{N}(A) = \mathcal{N}(A^+A) = \mathcal{R}(I_n - A^+A)$.

Ex5: $\mathcal{R}(A^*) = \mathcal{R}(A^+) = \mathcal{R}(A^+A) = \mathcal{N}(I_n - A^+A)$.
$\mathcal{N}(A^*) = \mathcal{N}(A^+) = \mathcal{N}(AA^+) = \mathcal{R}(I_m - AA^+)$.

Comment: The collection of all solutions to $Ax = 0$ is

$$\mathcal{N}(A) = \mathcal{N}(A^+A) = \mathcal{R}(I - A^+A) = \mathcal{R}(B)$$

where the columns of $B \in C^{m\times r}$ form a basis of $\mathcal{R}(I - A^+A)$ and $r = n - \mathrm{rank}(A)$.

Ex6: Consider $Ax = 0$ where $A = (1,\, 2)$.

$A = (1,\, 2) \Longrightarrow A^+ = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow A^+A = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \Longrightarrow I - A^+A = \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$.

$\mathcal{R}\left[\frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\right] = \mathcal{R}\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right] = \left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix} h : h \in C\right\}$ gives all solutions.

L16: Orthogonal complements, projections and projectin matrices

1. Orthogonal complement $\mathcal{R}^{\perp}(A)$

   (1) Inner product, norm and angle
   For $x,\, y \in C^m$, $\langle x,\, y\rangle = y^*x = \sum_i x_i\overline{y}_i$ is Frobenius inner product of x and y. With induced norm $\|x\| = \sqrt{\langle x,\, x\rangle}$, the angle formed by x and y is $\theta \in [0,\, \pi]$ such that $\cos(\theta) = \frac{\langle x,\, y\rangle}{\|x\|\cdot\|y\|}$. So x and y perpendicular$\Longleftrightarrow x \perp y \Longleftrightarrow \langle x,\, y\rangle = 0$.

   (2) $\mathcal{R}^{\perp}(A)$
   The collection of all z perpendicular to $\mathcal{R}(A)$ is the orthogonal complement of $\mathcal{R}(A)$ denoted as $\mathcal{R}^{\perp}(A)$

   $$\mathcal{R}^{\perp}(A) = \{z \in C^m :\, z \perp \mathcal{R}(A)\} = \{z \in C^m :\, \langle z,\, y\rangle = 0 \text{ for all } y \in \mathcal{R}(A)\}.$$

   (3) $\mathcal{R}^{\perp}(A)$ is a subspace of C^m.

   Proof If $u,\, v \in \mathcal{R}^{\perp}(A)$, then $\langle u,\, y\rangle = 0$ and $\langle v,\, y\rangle = 0$ for all $y \in \mathcal{R}(A)$.
   So $\langle \alpha u + \beta v,\, y\rangle = \alpha\langle u,\, y\rangle + \beta\langle v,\, y\rangle = 0$. Thus $\alpha u + \beta v \in \mathcal{R}^{\perp}(A)$.
   Therefore $\mathcal{R}^{\perp}(A)$ is a subspace of C^m.

   (4) $\mathcal{R}^{\perp}(A) = \mathcal{N}(A^*)$.
   Proof $$\begin{aligned} z \in \mathcal{R}^{\perp}(A) &\Longleftrightarrow z \perp \mathcal{R}(A) \Longleftrightarrow z \perp y \text{ for all } y \in \mathcal{R}(A) \\ &\Longleftrightarrow z \perp Ax \text{ for all } x \in C^n \Longleftrightarrow \langle z,\, Ax\rangle = 0 \text{ for all } x \\ &\Longleftrightarrow x^*A^*z = 0 \text{ for all } x \Longleftrightarrow A^*z = 0 \Longleftrightarrow z \in \mathcal{N}(A^*). \end{aligned}$$

   (5) $\mathcal{R}^{\perp}(A) = \mathcal{N}(A^*) = \mathcal{N}(A^+) = \mathcal{N}(AA^+) = \mathcal{R}(I - AA^+)$.

   Ex1: $\mathcal{R}^{\perp}(A^*) = \mathcal{N}(A)$; $\quad \mathcal{R}^{\perp}(A^+) = \mathcal{N}((A^+)^*) = \mathcal{N}(A)$
   So $\mathcal{R}^{\perp}(A^*) = \mathcal{R}^{\perp}(A^+) = \mathcal{N}(A) = \mathcal{N}(A^+A) = \mathcal{R}(I - A^+A)$.

2. Orthogonal complement $\mathcal{N}^{\perp}(A)$

   (1) Definition of $\mathcal{N}^{\perp}(A)$
   The orthogonal complement of $\mathcal{N}(A)$ is

   $$\mathcal{N}^{\perp}(A) = \{z \in C^n :\, z \perp \mathcal{N}(A)\} = \{z \in C^n :\, z \perp x \text{ for all } x \in \mathcal{N}(A)\}.$$

   (2) $\mathcal{N}^{\perp}(A) = \mathcal{R}^{\perp}(I - A^+A)$
   Proof $$\begin{aligned} \mathcal{N}^{\perp}(A) &= \{z \in C^n :\, z \perp \mathcal{N}(A)\} \\ &= \{z \in C^n\, z \perp \mathcal{R}(I - A^+A)\} = \mathcal{R}^{\perp}(I - A^+A). \end{aligned}$$

   (3) $\mathcal{N}^{\perp}(A) = \mathcal{R}(A^*)$
   Proof $\mathcal{N}^{\perp}(A) = \mathcal{R}^{\perp}(I - A^+A) = \mathcal{N}(I - A^+A) = \mathcal{R}(A^+A) = \mathcal{R}(A^*)$.

   (4) $\mathcal{N}^{\perp}(A) = \mathcal{R}^{\perp}(I - A^+A) = \mathcal{N}(I - A^+A) = \mathcal{R}(A^+A)$

   Ex2: $\mathcal{N}^{\perp}(A^*) = \mathcal{R}(A)$; $\quad \mathcal{N}^{\perp}(A^+) = \mathcal{R}((A^+)^*) = \mathcal{R}(A)$
   So $\mathcal{N}^{\perp}(A^*) = \mathcal{N}^{\perp}(A^+) = \mathcal{R}(A) = \mathcal{N}(I - AA^+) = \mathcal{R}^{\perp}(I - AA^+)$.

   Ex3: Show that Ax and $(I - AA^+)y$ are perpendicular.

   (i) Direct computation:
   $\langle Ax,\, (I - AA^+)y\rangle = [(I - AA^+)y]^*Ax = y^*(I - AA^+)Ax = y^*(A - A)x = 0.$

(ii) By concepts:
$Ax \in \mathcal{R}(A)$ and $(I - AA^+)y \in \mathcal{R}(I - AA^+) = \mathcal{N}(AA^+) = \mathcal{N}(A^*) = \mathcal{R}^\perp(A)$.
So $Ax \perp (I - AA^+)y$.

3. Projections and projection matrices

(1) Pythagorean theorem
If $x \perp y$, then $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$.
Proof $\|x \pm y\|^2 = \langle x \pm y,\, x \pm y\rangle = \langle x,\, x\rangle \pm \langle x,\, y\rangle \pm \langle y,\, x\rangle + \langle y,\, y\rangle$
$= \|x\|^2 \pm 0 \pm 0 + \|y\|^2 = \|x\|^2 + \|y\|^2$.

(2) Projection
S is a subspace in C^k, for $y \in C^k$ there exists a unique $\widehat{y} \in S$ such that

$$\|y - \widehat{y}\|^2 \leq \|y - z\|^2 \text{ for all } z \in S.$$

This $\widehat{y}$ is the projection of y onto S denoted as $\pi(y|S)$.
Thus $\pi(y|\mathcal{R}(A))$, $\pi(y|\mathcal{R}^\perp(A))$, $\pi(x|\mathcal{N}(A))$ and $\pi(x|\mathcal{N}^\perp(A))$ are all defined.

(3) Lemma
If $\widehat{y} \in S$ and $y - \widehat{y} \in S^\perp$, then $\widehat{y} = \pi(y|S)$.
Proof $\widehat{y} \in S$. For $z \in S$, with $y - \widehat{y} \in S^\perp$ and $\widehat{y} - z \in S$, $(y - \widehat{y}) \perp (\widehat{y} - z)$.
By Pythagorean theorem

$$\|y - z\|^2 = \|(y - \widehat{y}) + (\widehat{y} - z)\|^2 = \|y - \widehat{y}\|^2 + \|\widehat{y} - z\|^2 \geq \|y - \widehat{y}\|^2.$$

So $\widehat{y} = \pi(y|S)$.

(4) $\pi(y|\mathcal{R}(A)) = AA^+y$. AA^+ is the projection matrix onto $\mathcal{R}(AA^+) = \mathcal{R}(A)$.
Proof $AA^+y \in \mathcal{R}(AA^+) = \mathcal{R}(A)$.
$y - AA^+y = (I - AA^+)y \in \mathcal{R}(I - AA^+) = \mathcal{N}(AA^+) = \mathcal{N}(A^+) = \mathcal{R}^\perp(A)$.
By Lemma in (3), $\pi(y|\mathcal{R}(A)) = AA^+y$.
Comment: AA^+ is called the projection matrix onto $\mathcal{R}(AA^+) = \mathcal{R}(A)$.

(5) Projections onto other spaces
$\pi(y|\mathcal{R}^\perp(A)) = \pi(y|\mathcal{R}(I - AA^+)) = (I - AA^+)y$.
$I - AA^+$ is the projection matrix onto $\mathcal{R}(I - AA^+) = \mathcal{R}^\perp(A)$.
$\pi(x|\mathcal{N}(A)) = \pi(x|\mathcal{R}(I - A^+A)) = (I - A^+A)x$.
$I - A^+A$ is the projection matrix onto $\mathcal{R}(I - A^+A) = \mathcal{N}(A)$.
$\pi(x|\mathcal{N}^\perp(A)) = \pi(x|\mathcal{R}(A^+A)) = A^+Ax$.
A^+A is the projection matrix onto $\mathcal{R}(A^+A) = \mathcal{N}^\perp(A)$.

Comment: For given $x_0 \in C^n$ and $A \in C^{m\times n}$, the collection of all solutions to $Ax = 0$ is $\mathcal{N}(A)$. Among all solutions, the one with minimum distance to x_0 is $\pi(x_0|\mathcal{N}(A))$.

$$\pi(x_0|\mathcal{N}(A)) = \pi(x_0|\mathcal{R}(I - A^+A)) = (I - A^+A)x_0.$$

Ex4: With $A = (1,\, 2)$ and $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = (1,\, 2) \Longrightarrow A^+ = \tfrac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow A^+A = \tfrac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \Longrightarrow I - A^+A = \tfrac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

$$(I - A^+A)x_0 = \tfrac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$