

L15: $\mathcal{R}(A)$, $\mathcal{N}(A)$ and their relations

1. $\mathcal{R}(A)$

(1) Definition and notations

For $A \in C^{m \times n}$, $\{Ax : x \in C^n\} \subset C^m$
 is a linear space in C^m denoted as $L(A)$; It is the range of the linear transformation (LT) $y = Ax$ and is denoted as $\mathcal{R}(A)$; It is the column space of A and is denoted as $\mathcal{C}(A)$; As the span of the columns of A it is also denoted as $\mathcal{S}(A)$.
 $\dim[\mathcal{R}(A)] = \text{rank}(A)$.

Ex1: In linear regression $y = X\beta + \epsilon$ with $E(\epsilon) = 0$, $E(y) = X\beta$. So $E(y) \in \mathcal{R}(X)$.

(2) A lemma: $\mathcal{R}(AB) \subset \mathcal{R}(A)$.

Proof: $y \in \mathcal{R}(AB) \implies y = ABx = A(Bx) \in \mathcal{R}(A)$. So $\mathcal{R}(AB) \subset \mathcal{R}(A)$. $\square$

(3) Expressions

$\mathcal{R}(A) = \mathcal{R}(AA^-) = \mathcal{R}(AA^+) = \mathcal{R}(AA^*) = \mathcal{R}((A^*)^+)$.

Proof: $\mathcal{R}(AA^-) \subset \mathcal{R}(A) = \mathcal{R}(AA^-A) \subset \mathcal{R}(AA^-) \implies \mathcal{R}(A) = \mathcal{R}(AA^-)$;

$\mathcal{R}(A) = \mathcal{R}(AA^-) \implies \mathcal{R}(A) = \mathcal{R}(AA^+)$;

$\mathcal{R}(AA^*) \subset \mathcal{R}(A) = \mathcal{R}(AA^+A) = \mathcal{R}(AA^*(A^+)^*) \subset \mathcal{R}(AA^*) \implies \mathcal{R}(A) = \mathcal{R}(AA^*)$;

$\mathcal{R}(A) = \mathcal{R}(AA^+A) = \mathcal{R}((A^+)^*A^*A) \subset \mathcal{R}((A^+)^*) = \mathcal{R}((A^+AA^+)^*)$

$= \mathcal{R}(AA^+(A^+)^*) \subset \mathcal{R}(A)$. $\implies \mathcal{R}(A) = \mathcal{R}((A^+)^*)$.

So $\mathcal{R}(A) = \mathcal{R}(AA^-) = \mathcal{R}(AA^+) = \mathcal{R}(AA^*) = \mathcal{R}((A^*)^+)$. $\square$

Comment: AA^* is Hermitian; AA^- is idempotent; AA^+ is Hermitian and idempotent.

Ex2: $\mathcal{R}(A^*) = \mathcal{R}(A^*(A^*)^+) = \mathcal{R}(A^+A)$; $\mathcal{R}(A^*) = \mathcal{R}(A^*(A^*)^*) = \mathcal{R}(A^*A)$;

$\mathcal{R}(A^*) = \mathcal{R}(A^{**})^+ = \mathcal{R}(A^+)$. So $\mathcal{R}(A^*) = \mathcal{R}(A^+) = \mathcal{R}(A^+A)$.

But $\mathcal{R}(A^*) \neq \mathcal{R}(A^-A)$.

2. $\mathcal{N}(A)$

(1) Definition and notation

For $A \in C^{m \times n}$, $\{x \in C^n : Ax = 0\} \subset C^n$
 is a subspace of C^n . This space is the Kernel of the LT $y = Ax$, the solution space for the equation $Ax = 0$ called the null space of A denoted as $\mathcal{N}(A)$. $\dim[\mathcal{N}(A)] = n - \text{rank}(A)$.

(2) A lemma: $\mathcal{N}(A) \subset \mathcal{N}(BA)$.

Proof: $x \in \mathcal{N}(A) \implies Ax = 0 \implies BAx = 0 \implies x \in \mathcal{N}(BA)$. $\square$

(3) Expressions

$\mathcal{N}(A) = \mathcal{N}(A^-A) = \mathcal{N}(A^+A) = \mathcal{N}(A^*A) = \mathcal{N}((A^*)^+)$.

Proof: $\mathcal{N}(A^-A) \subset \mathcal{N}(AA^-A) = \mathcal{N}(A) \subset \mathcal{N}(A^-A) \implies \mathcal{N}(A) = \mathcal{N}(A^-A)$;

$\mathcal{N}(A) = \mathcal{N}(A^-A) \implies \mathcal{N}(A) = \mathcal{N}(A^+A)$;

$\mathcal{N}(A^*A) \subset \mathcal{N}((A^+)^*A^*A) = \mathcal{N}(AA^+A) = \mathcal{N}(A) \subset \mathcal{N}(A^*A) \implies \mathcal{N}(A) = \mathcal{N}(A^*A)$

$\mathcal{N}((A^+)^*) \subset \mathcal{N}(AA^*(A^+)^+) = \mathcal{N}(AA^+A) = \mathcal{N}(A) \subset \mathcal{N}((A^+)^*A^+A)$
 $= \mathcal{N}((A^*)^+A^*(A^+)^+) = \mathcal{N}((A^+)^*) \implies \mathcal{N}(A) = \mathcal{N}((A^*)^+)$.

So $\mathcal{N}(A) = \mathcal{N}(A^-A) = \mathcal{N}(A^+A) = \mathcal{N}(A^*A) = \mathcal{N}((A^*)^+)$. $\square$

Comment: A^*A is Hermitian; A^-A is idempotent; A^+A is Hermitian and idempotent.

Ex3: $\mathcal{N}(A^*) = \mathcal{N}((A^*)^+A^*) = \mathcal{N}(AA^+)$; $\mathcal{N}(A^*) = \mathcal{N}((A^*)^*A^*) = \mathcal{N}(AA^*)$;

$\mathcal{N}(A^*) = \mathcal{N}((A^{**})^+) = \mathcal{N}(A^+)$. So $\mathcal{N}(A^*) = \mathcal{N}(A^+) = \mathcal{N}(AA^+)$.

But $\mathcal{N}(A^*) \neq \mathcal{N}(AA^-)$.

3. Cross expressions

(1) A lemma

Suppose D is idempotent. Then (i) $\mathcal{R}(D) = \mathcal{N}(I - D)$, (ii) $\mathcal{N}(D) = \mathcal{R}(I - D)$.

Proof: (i) $y \in \mathcal{R}(D) \implies y = Dx \implies (I - D)y = 0 \implies y \in \mathcal{N}(I - D)$.

$x \in \mathcal{N}(I - D) \implies (I - D)x = 0 \implies x = Dx \in \mathcal{R}(D)$.

So $\mathcal{R}(D) = \mathcal{N}(I - D)$.

(ii) When D is idempotent, $I - D$ is idempotent. By (i)

$\mathcal{R}(I - D) = \mathcal{N}(I - (I - D)) = \mathcal{N}(D)$, i.e., $\mathcal{N}(D) = \mathcal{R}(I - D)$. $\square$

(2) Express $\mathcal{R}(\cdot)$ by $\mathcal{N}(\cdot)$

$$\mathcal{R}(A) = \mathcal{R}(AA^-) = \mathcal{N}(I_m - AA^-) \quad \mathcal{R}(A) = \mathcal{R}(AA^+) = \mathcal{N}(I_m - AA^+).$$

(3) Express $\mathcal{N}(\cdot)$ by $\mathcal{R}(\cdot)$

$$\mathcal{N}(A) = \mathcal{N}(A^-A) = \mathcal{R}(I_n - A^-A) \quad \mathcal{N}(A) = \mathcal{N}(A^+A) = \mathcal{R}(I_n - A^+A).$$

Ex4: $\mathcal{R}(A^*) = \mathcal{R}(A^+) = \mathcal{R}(A^+A) = \mathcal{N}(I_n - A^+A)$.

$$\mathcal{N}(A^*) = \mathcal{N}(A^+) = \mathcal{N}(AA^+) = \mathcal{R}(I_m - AA^+).$$

Ex5: Find all solutions to $Ax = 0$

$$Ax = 0 \iff x \in \mathcal{N}(A) = \mathcal{R}(I - A^+A) = \{(I - A^+A)u : u \in C^m\}.$$

Thus $\{(I - A^+A)u : u \in C^m\}$ is the collection of all solutions to $Ax = 0$.