L13 Moore-Penrose inverses

1. Moore-Penrose inverse

   (1) Moore-Penrose inverse
   For $A \in C^{m\times n}$ matrix $G \in C^{n\times m}$ satisfying the following four Penrose conditions exists, is unique, and is called the Moore-Penrose inverse of A denoted by A^+.

   $$\text{(i) } AGA = A \qquad \text{(ii) } GAG = G \qquad \text{(iii) } (AG)^* = GB \qquad \text{(iv) } (GA)^* = GA.$$

   Proof Suppose by SVD $A = U\begin{pmatrix}\Delta & 0\\ 0 & 0\end{pmatrix}V^*$. We show that

   $$G \text{ satisfying the four Penrose conditions } \Longleftrightarrow G = V\begin{pmatrix}\Delta^{-1} & 0\\ 0 & 0\end{pmatrix}U^*.$$

   ⇒: (i) $AGA = A \Longrightarrow G \in A^- \Longrightarrow G = V\begin{pmatrix}\Delta^{-1} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}U^*$.
   (iii) AG is Hermitian $\Longrightarrow U\begin{pmatrix}I_r & \Delta H_{12}\\ 0 & 0\end{pmatrix}U^*$ is Hermitian. So $H_{12} = 0$.
   (iv) GA is Hermitian $\Longrightarrow V\begin{pmatrix}I_r & 0\\ H_{21}\Delta & 0\end{pmatrix}V^*$ is Hermitian. So $H_{21} = 0$.
   (ii) $GAG = G \Longrightarrow H_{22} = 0$. Hence $G = V\begin{pmatrix}\Delta^{-1} & 0\\ 0 & 0\end{pmatrix}U^*$.

   ⇐: (i) $AGA = U\begin{pmatrix}\Delta & 0\\ 0 & 0\end{pmatrix}V^* = A$. (ii) $GAG = V\begin{pmatrix}\Delta^{-1} & 0\\ 0 & 0\end{pmatrix}U^* = G$.
   (iii) $AG = U\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}U^*$ is Hermitian. (iv) $GA = V\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}V^*$ is Hermitian.
   Hence G satisfying the four Penrose conditions.

   Comments: $A^+ \in A^-$. For SVD $A = U\begin{pmatrix}\Delta & 0\\ 0 & 0\end{pmatrix}V^*$, $A^+ = V\begin{pmatrix}\Delta^{-1} & 0\\ 0 & 0\end{pmatrix}U^*$ is SVD for A^+. $\operatorname{rank}(A^+) = \operatorname{rank}(A)$.

   (2) $0^+_{m\times n} = 0_{n\times m}$; With $\alpha \neq 0$, $(\alpha A)^+ = \frac{1}{\alpha}A^+$
   Proof For the first one let $G = 0_{n\times m}$. Check (iii) only.
   (iii) $0_{m\times n}G = 0_{m\times m}$ is real symmetric and hence is Hermitian;
   For the second one let $G = \frac{1}{\alpha}A^+$. Check (iv) only.
   (iv) $G(\alpha A) = \frac{1}{\alpha}A^+\alpha A = A^+A$ is Hermitian.

   (3) $(A')^+ = (A^+)'$, $(\overline{A})^+ = \overline{A^+}$, $(A^*)^+ = (A^+)^*$ and $(A^+)^+ = A$
   Proof For the 1st one let $G = (A^+)'$. Check (i): $A'GA' = A'(A^+)'A' = (AA^+A)' = A'$.
   For the 2nd one let $G = \overline{A^+}$. Check (ii): $G\overline{A}G = \overline{A^+}\,\overline{A}\,\overline{A^+} = \overline{A^+AA^+} = \overline{A^+}$.
   For the 3rd one let $G = (A^+)^*$. Check (iii): $A^*G = A^*(A^+)^* = (A^+A)^*$ is Hermitian.
   For the last one let $G = A$. Check (iv): $GA^+ = AA^+$ is Hermitian.

Ex1: For column vector $0 \neq x \in C^n$, $x^+ = \frac{x^*}{\|x\|^2}$.
Check (i) and (ii). (i): $x\frac{x^*}{\|x\|^2}x = x$, (ii) $\frac{x^*}{\|x\|^2}x\frac{x^*}{\|x\|^2} = \frac{x^*}{\|x\|^2}$.

Comment: $(x')^+ = (x^+)' = \frac{(x')^*}{\|x\|^2}$.

2. More special cases

(1) If $A \in C^{m\times n}$ has full row rank, then $AA^* \in C^{m\times m}$ is non-singular.
Then $A^+ = A^*(AA^*)^{-1}$.

Proof Let $G = A^*(AA^*)^{-1}$. Only check (i).
(i) $AGA = A[A^*(AA^*)^{-1}]A = A$.

(2) If $A \in C^{m\times n}$ has full column rank, then $A^*A \in C^{n\times n}$ is non-singular.
Then $A^+ = (A^*A)^{-1}A^*$.

Proof Let $G = (A^*A)^{-1}A^*$. Only check (ii).
(ii) $GAG = [(A^*A)^{-1}A^*]A[(A^*A)^{-1}A^*] = (A^*A)^{-1}A^* = G$.

(3) If A has orthonormal columns, i.e., $A^*A = I$, then $A^+ = A^*$

Proof Let $G = A^*$. Only check iii). (iii) $AG = AA^*$ is Hermitian.

(4) If A has orthonormal rows, i.e., $(A')^*A' = I \iff AA^* = I$, then $A^+ = A^*$.

Proof Let $G = A^*$. Only check (iv). (iv) $GA = A^*A$ is Hermitian.

(5) If $A^* = A = A^2$, then $A^+ = A$.

Proof Let $G = A$. Only check (i). (i) $AGA = AAA = A$.

3. Special cases of $(AB)^+ = B^+A^+$

(1) $(AB)^+ \neq B^+A^+$.
For $A = (1, 2)$ and $B = \begin{pmatrix}1\\0\end{pmatrix}$, $(AB)^+ = 1^+ = 1$. But $B^+A^+ = (1, 0)\begin{pmatrix}1\\2\end{pmatrix}/5 = \frac{1}{5}$.

(2) If $AB = 0$, then $(AB)^+ = B^+A^+$.

Proof If $AB = 0$, then $(AB)^+ = 0^+ = 0$. On the other hand,
$B^+A^+ = B^+BB^+A^+AA^+ = B^+(B^+)^*(AB)^*(A^+)^*A^* = 0$.

(3) If A has orthonormal columns, i.e., $A^*A = I$, then $(AB)^+ = B^+A^+$

Proof Let $G = B^+A^+ = BA^*$. Only check (iii).
(iii) $ABG = ABB^+A^* = A(BB^+)A^*$ is Hermitian.

(4) If B has orthonormal rows, i.e., $BB^* = I$, then $(AB)^+ = B^+A^+$

Proof Let $G = B^+A^+ = B^*A^+$. Only check (ii).
(ii) $GABG = (B^*A^+)AB(B^*A^+) = B^*A^+AA^+ = B^+A^+ = G$.

(5) If A has full column rank and B has full row rank, then $(AB)^+ = B^+A^+$.

Proof A has full column rank $\Longrightarrow A^+ \in A^- = A^L \Longrightarrow A^+A = I$.
B has full row rank $\Longrightarrow B^+ \in B^- = B^R \Longrightarrow BB^+ = I$.
Let $G = B^+A^+$. Only check (i).
(i) $(AB)G(AB) = (AB)(B^+A^+)(AB) = A(BB^+)(A^+A)B = IIB = AB$.

Ex2: Suppose for A both G_1 and G_2 satisfy the four Penrose conditions. Show $G_1 = G_2$.

$$
\begin{aligned}
G_1 &\overset{(ii)}{=\!=} G_1AG_1 \overset{(i)}{=\!=} G_1(AG_2A)G_1 \overset{(iii)}{=\!=} G_1(AG_2)^*(AG_1)^* = G_1G_2^*A^*G_1^*A^* \\
&= G_1G_2^*(AG_1A)^* \overset{(i)}{=\!=} G_1G_2^*A^* = G_1(AG_2)^* \overset{(iii)}{=\!=} G_1AG_2 \overset{(i)}{=\!=} G_1(AG_2A)G_2 \\
&\overset{(iv)}{=\!=} (G_1A)^*(G_2A)^*G_2 = A^*G_1^*A^*G_2^*G_2 = (AG_1A)^*G_2^*G_2 \overset{(i)}{=\!=} A^*G_2^*G_2 = (G_2A)^*G_2 \\
&\overset{(iv)}{=\!=} G_1AG_2 \overset{(ii)}{=\!=} G_2.
\end{aligned}
$$

L14 More on Moore-Penrose inverses

1. AA^* and A^*A

   (1) $(AA^*)^+ = (A^*)^+A^+$ and $(A^*A)^+ = A^+(A^*)^+$

   Proof For the first one let $G = (A^*)^+A^+$. Check (i) only.

   $$\begin{aligned}\text{(i) } (AA^*)G(AA^*) &= (AA^*)[(A^*)^+A^+](AA^*) = A[A^*(A^*)+](A^+A)A^* \\ &= A[A^*(A^*)^+]^*(A^+A)^*A^* = (AA^+A)(AA^+A)^* = AA^*.\end{aligned}$$

   (2) $A^*(AA^*)^+ = A^+ = (A^*A)^+A^*$.

   Comment: Recall: A has full row rank $\Longrightarrow A^*(AA^*)^{-1} = A^+$
   A has full column rank $\Longrightarrow (AA^*)^{-1}A^* = A^+$.
   (2) gives what if the conditions are removed.

   Proof Show 1st one only: $A^*(AA^*)^+ = A^*(A^*)^+A^+ = (A^+A)^*A^+ = A^+AA^+ = A^+$.

   (3) $A(A^*A)^-A^* = AA^+$ and $A^*(AA^*)^-A = A^+A$.

   Comment: While $(A^*A)^+A^* = A^+$ and $A^*(AA^*)^+ = A^+$, non-unique $(A^*A)^-A^* \neq A^+$ and non-unique $A^*(AA^*)^- \neq A^+$. However (3) holds.

   Proof Show the first one only.

   $$\begin{aligned}A(A^*A)^-A^* &= AA^+A(A^*A)^-(AA^+A)^* = (AA^+)^*A(A^*A)^-A^*(AA^+)^* \\ &= (A^+)^*A^*A(A^*A)^-A^*AA^+ = (A^+)^*A^*AA^+ = (AA^+)^*AA^+ \\ &= AA^+.\end{aligned}$$

2. Hermitian and idempotent matrices

   (1) Recall: if $P^* = P = P^2$, then $P^+ = P$. So $PP^+ = P$ and $P^+P = P$.

   Comment: AA^+, A^+A, $I - AA^+$ and $I - A^+A$ are all Hermitian and idempotent, and hence satisfy the conditions for P.

   (2) The following (a), (b), (c) and (d) are equal.
   (a) $(ADB)^+$ (b) $B^+B(ADB)^+$ (c) $(ADB)^+AA^+$ (d) $B^+B(ADB)^+AA^+$

   Proof For (a)=(b) let $G = B^+B(ADB)^+$. Show (i) only.

   $$\begin{aligned}\text{(i)} \quad (ADB)G(ADB) &= (ADB)[B^+B(ADB)^+](ADB) \\ &= (ADB)(ADB)^+(ADB) = ADB\end{aligned}$$

   For (a)=(c) let $G = (ADB)^+AA^+$. Show (ii) only.

   $$\begin{aligned}\text{(ii)} \quad G(ADB)G &= [(ADB)^+AA^+](ADB)[(ADB)^+AA+] \\ &= (ADB)^+(ADB)(ADB)^+AA^+ = G.\end{aligned}$$

   For (a)=(d) let $G = B^+B(ADB)^+AA^+$. Show (iii) only.

   $$\begin{aligned}\text{(iii)} \quad (ADB)G &= (ADB)[B^+B(ADB)^+AA^+] = (ADB)(ADB)^+AA^+ \\ &= [AA^+(ADB)(ADB)^+]^* = (ADB)(ADB)^+ \text{ is Hermitian.}\end{aligned}$$

   Ex1: $(AB)^+ = B^+B(AB)^+ = (AB)^+AA^+ = B^+B(AB)^+AA^+$.

   Ex2: $(I - B^+B)(AB)^+ = 0$ since $(I - B^+B)(AB)^+ = (I - B^+B)B^+B(AB)^+ = 0$.
   $[(I - AA^+)B]^+A = 0$ since $[(I - AA^+)B]^+A = [(I - AA^+)B]^+(I - AA^+)A = 0$.

   Ex3: $(AB)(AB)^+AA^+ = (AB)(AB)^+$ is directly by Ex1 or

   $$\begin{aligned}(AB)(AB)^+AA^+ &= \{[(AB)(AB)^+(AA^+)]^*\}^* = [(AA^+)(AB)(AB)^+]^* \\ &= [(AB)(AB)^+]^* = (AB)(AB)^+\end{aligned}$$

3. Matrices with blocks

(1) In $(A,\, B)$

$$\begin{aligned} & \text{The columns of } A \text{ and the columns of } B \text{ are orthogonal} \\ \Longleftrightarrow\ & A^*B = 0 \Longleftrightarrow B^*A = 0 \Longleftrightarrow B^+A = 0 \Longleftrightarrow A^+B = 0. \end{aligned}$$

In $\begin{pmatrix} A \\ B \end{pmatrix}$,

$$\begin{aligned} & \text{The rows of } A \text{ and the rows of } B \text{ are orthogonal} \\ \Longleftrightarrow\ & AB^* = 0 \Longleftrightarrow BA^* = 0 \Longleftrightarrow BA^+ = 0 \Longleftrightarrow AB^+ = 0. \end{aligned}$$

(2) $(A,\, B)^+ = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} \Longleftrightarrow$ The columns of A and the columns of B are orthogonal.

Proof $\Rightarrow$**:** With $(A,\, B)^+ = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix}$, $(A,\, B)\begin{pmatrix} A^+ \\ B^+ \end{pmatrix}(A,\, B) = (A,\, B)$

$\Longrightarrow (A + BB^+A,\, AA^+B + B) = (A,\, B) \Longrightarrow BB^+A = 0 \Longrightarrow B^+A = 0.$

$\Leftarrow$**:** For $(A,\, B)^+ = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix}$, we check (i) only.

(i) $(A,\, B)\begin{pmatrix} A^+ \\ B^+ \end{pmatrix}(A,\, B) = (AA^+ + BB^+)(A,\, B) = (A,\, B).$

Comment: If in $A = (A_1, .., A_k)$ $A_i^*A_j = 0$ for all $i \neq j$, then $A^+ = \begin{pmatrix} A_1^+ \\ \vdots \\ A_k^+ \end{pmatrix}$.

(3) $\begin{pmatrix} A \\ B \end{pmatrix}^+ = (A^+,\, B^+) \Longleftrightarrow$ The rows of A and B are orthogonal.

Proof Skipped.

Comment: If in $A = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix}$, $A_iA_j^* = 0$ for all $i \neq j$, then $A^+ = (A_1^+, .., A_k^+)$.

Ex4: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^+ = \begin{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}^+ \\ \begin{pmatrix} 0 \\ B \end{pmatrix}^+ \end{pmatrix} = \begin{pmatrix} A^+ & 0 \\ 0 & B^+ \end{pmatrix}$

since $\begin{pmatrix} A \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ B \end{pmatrix} = 0$, $A0^* = 0$ and $0B^* = 0$.

$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^+ = ((A,\, 0)^+,\, (0,\, B)^+) = \begin{pmatrix} A^+ & 0 \\ 0 & B^+ \end{pmatrix}$

since $(A,\, 0)(0,\, B)^* = 0$, $A^*0 = 0$ and $0^*B = 0$.

Ex5: $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^+ =?$

Stat701 **Exercise**

1. 4.1 p191. Find the singular value matrix Δ for $A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

2. 5.2 p238. Find A^+ for $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ using Theorem 5.3 (h) $A^+ = (A'A)^{-1}A'$.
   Hint: Calculation tool for $M^{-1}H$: $(M|H) \xrightarrow{row} (I|M^{-1}H)$

3. 5.3 p238. Find $\mathbf{a}^+$ for $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}$.

4. 5.12 (a) p239. For real matrix A show $A'AA^+ = A'$ and $A^+AA' = A'$ separately.

5. 5.16 p240. B has full row rank. Show $(AB)(AB)^+ = AA^+$.
   Hint: first show $[AA^+(AB)(AB)^+]^* = (AB)(AB)^+$.
   Next, show $[AA^+(AB)(AB)^+]^* = AA^+$ under the condition that B has a right-inverse B^R.