L09 Diagonalizable matrices

1. Similar matrices

   (1) Definition
   A is similar to $B \overset{def}{\Longleftrightarrow}$ There exists X such that $A = XBX^{-1}$.
   A, B, X are all $n \times n$ matrices and X is non-singular.

   (2) "Similar to" is a relation of equivalence

   (i) Reflexivity: A is similar to A since $A = IAI^{-1}$

   (ii) Symmetry: A is similar to $B \Longleftrightarrow B$ is similar to A
   " $\Rightarrow$ " only: A is similar to $B \Longrightarrow A = XBX^{-1} \Longrightarrow B = X^{-1}A(X^{-1})^{-1}$
   $\Longrightarrow B$ is similar to A.

   (iii) Transitivity: If A is similar to B and B is similar to C, then A is similar to C.
   $A = XBX^{-1}$ and $B = YCY^{-1} \Longrightarrow A = (XY)C(XY)^{-1}$.

   (3) Properties
   If A and B are similar, i.e., $A = XBX^{-1}$, then A, B share ranks, determinants, trace, characteristic polynomials and hence eigenvalues.

   Pf: $\text{rank}(A) = \text{rank}(XBX^{-1}) = \text{rank}(B)$. $|A| = |XBX^{-1}| = |X|\,|B|\,|X^{-1}| = |B|$.
   $\text{tr}(A) = \text{tr}(XBX^{-1}) = \text{tr}(BX^{-1}X) = \text{tr}(B)$.
   $|A - \lambda I| = |XBX^{-1} - \lambda XX^{-1}| = |X|\,|B - \lambda I|\,|X^{-1}| = |B - \lambda I|$.

   Ex1: If $A = XBX^{-1}$, with a polynomial $p(t)$, $p(A) = Xp(B)X^{-1}$.
   For example with $p(t) = 3t^2 - 2t + 4$,
   $$\begin{aligned} p(A) &= 3A^2 - 2A + 4I = 3(XBX^{-1})(XBX^{-1}) - 2(XBX^{-1}) + 4XX^{-1} \\ &= X(3B^2 - 2B + 4I)X^{-1} = Xp(B)X^{-1}. \end{aligned}$$

2. Diagonalizable matrices

   (1) Diagonal matrix
   Let $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$. Then the eigenvalues of Λ are $\lambda_1, ..., \lambda_n$;
   $\text{rank}(\Lambda) = \#$ of non-zero λ_i; $\det(\Lambda) = \lambda_1 \cdots \lambda_n$; $\text{tr}(\Lambda) = \lambda_1 + \cdots + \lambda_n$;
   With polynomial $p(\cdot)$, $p(\Lambda) = \text{diag}(p(\lambda_1), ..., p(\lambda_n))$.

   Ex2: With $\Lambda = \text{diag}(\lambda_1,\, \lambda_2)$, $\Lambda^2 - 2 = \text{diag}(\lambda_1^2 - 2,\, \lambda_2^2 - 2)$.

   (2) Diagonalizable matrices
   A is diagonalizable$\overset{def}{\Longleftrightarrow} A$ is similar to a diagonal matrix

   (3) Properties
   If A is similar to Λ, i.e., $A = X\Lambda X^{-1}$, then the eigenvalues of A are $\lambda_1, ..., \lambda_n$;
   $\text{rank}(A) = \#$ of non-zero λ_i; $\det(A) = \lambda_1 \cdots \lambda_n$; $p(A) = X\text{diag}(p(\lambda_1), ..., p(\lambda_n))X^{-1}$.

   (4) Sufficient and necessary condition for A to be diagonalizable

   $$A \in C^{n\times n} \text{ is diagonalizable } \Longleftrightarrow A \in C^{n\times n} \text{ has } n \text{ LI eigenvectors.}$$

   Proof. $\Rightarrow$**:** If $A = X\Lambda X^{-1}$, then $AX = X\Lambda$ and $|X| \neq 0$.
   So the columns of X are n LI eigenvectors of A.

   $\Leftarrow$**:** Suppose $Ax_i = \lambda_i x_i$, $x_i \neq 0$ and $x_1, ..., x_n$ are LI.
   Let $X = (x_1, ..., x_n)$ and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$.
   Then $AX = X\Lambda$ and X is non-singular. Therefore $A = X\Lambda X^{-1}$.

(5) More iff-conditions
A is diagonlizable $\Longleftrightarrow d_i = r_i$ for all $i \Longleftrightarrow d_i = r_i$ for all $r_i > 1$.

3. Schur-decomposition

(1) Normal matrices
$A \in C^{n\times n}$ is a normal matrix if $A = U\Lambda U^*$ where U is unitary and Λ is diagonal. Clearly, if A is normal, then A is diagonalizable. So Let $\mathcal{N}$ and $\mathcal{D}$ be the collections of all normal matrices and diagonalizable matrices respectively. Then

$$\mathcal{N} \subset \mathcal{D} \subset C^{n\times n}.$$

(2) Schur-decomposition
For $A \in C^{n\times n}$ there exist unitary $U \in C^{n\times n}$ and upper-triangular $T \in C^{n\times n}$ such that

$$A = UTU^*.$$

Comment: Schur-decomposition is also called Schur-triangulation. With this decomposition we are short in evidence to call A normal.

Proof We show the decomposition by induction on n.
When $n = 1$, $A \in C^{1\times 1}$ is upper-triangular and $A = 1A1^*$. Decomposition holds.
Assume the decomposition is true when $n = k$. Consider $A \in C^{(k+1)\times(k+1)}$.
Suppose A has eigenvalue λ with a unit eigenvector y. Let $Y = (y,\, Y_2)$ be unitary.

$$\begin{aligned} A &= YY^*AYY^* = Y\begin{pmatrix} y^* \\ Y_2^* \end{pmatrix}(\lambda y,\, AY_2)Y^* = Y\begin{pmatrix} \lambda & y^*AY_2 \\ 0 & Y_2^*AY_2 \end{pmatrix}Y^* \\ &\overset{*}{=} Y\begin{pmatrix} \lambda & y^*AY_2 \\ 0 & U_1T_1U_1^* \end{pmatrix}Y^* = Y\begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}\begin{pmatrix} \lambda & y^*AY_2U_1 \\ & \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & U_1^* \end{pmatrix}Y^* \\ &= UTU^*. \end{aligned}$$

$*$: For $Y_2^*AY_2 \in C^{k\times k}$, by induction assumption $Y_2^*AU_2 = U_1T_1U_1^*$ where T_1 is upper-triangular and U_1 is unitary.
$T = \begin{pmatrix} \lambda & y^*AY_2U_1 \\ 0 & T_1 \end{pmatrix}$ is upper-triangular and $U = Y\begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}$ is unitary since $UU^* = I_{k+1}$. Thus the decomposition holds for $n = k + 1$.

(3) Properties
By Schur decomposition, $A = UTU^*$, A and T are similar and hence A and T share eigenvalues, ranks, determinants, trace.
T has eigenvalues $t_{11}, ..., t_{nn}$; T has trace $t_{11} + \cdots + t_{11}$; T has determinants $t_{11}\cdots t_{nn}$; But the rank of T is not the number of non-zero diagonal elements.

Ex3: $T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is upper-triangular with one non-zero diagonal element. But rank$(T) = 2$.

L10 Normal matrices and Hermitian matrices

1. Normal matrices

   (1) Recall
   A is normal $\overset{def}{\Longleftrightarrow}$ $A = X\Lambda X^*$ where X is unitary and Λ is diagonal.

   (2) Iff-conditions via eigenvectors
   For $A \in C^{n\times n}$, the followings are equivalent

   (i) A is normal

   (ii) A has n orthonormal eigenvectors

   (iii) A has n orthogonal eigenvectors

   Proof (i)⇒(ii): If (i), then $A = X\Lambda X^*$ and $X^* = X^{-1}$. So $AX = X\Lambda$, $X^* = X^{-1}$. Thus the n columns of X are n orthonormal eigenvectors of A.

   (ii)⇒(iii): The n orthonormal eigenvectors are n orthogonal eigenvectors.

   (iii)⇒(i): Dividing each of n orthogonal eigenvectors by its norm we obtain n orthonormal eigenvectors, $x_1, ..., x_n$. Let $X = (x_1, ..., x_n)$. Then $AX = X\Lambda$, X is unitary. Hence $A = X\Lambda X^*$.

   Comment: A is normal if $d_i = r_i$ for all $i = 1, ..., k$, and $S_A(\lambda_i) \perp S_A(\lambda_j)$ for $\lambda_i \neq \lambda_j$.

   Ex: $\Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ with two simple eigenvalues 1 and 2 is diagonalizable.

   But $S_A(1) = \text{Span}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ and $S_A(2) = \left[\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right]$ are not perpendicular.

   So A does not have two orthogonal eigenvectors. Thus A is not normal.

   (3) A simple sufficient and necessary condition

$$A \text{ is a normal matrix} \Longleftrightarrow A^*A = AA^*.$$

   Pf: **⇒:** A is normal $\Longrightarrow A = X\Lambda X^* \Longrightarrow A^*A = X\Lambda^*\Lambda X^* = X\Lambda\Lambda^* X^* = AA^*$

   ⇐: By Schur decomposition $A = XTX^*$.

   $A^*A = AA^* \Longrightarrow TT^* = T^*T \overset{def}{=\!=} H \overset{**}{\Longrightarrow} T = \Lambda \Longrightarrow A = X\Lambda X^*$.

   $**$: Examining h_{ii}, $i = 1, ..., n$, leads to $H = \Lambda$. For example consider h_{11} in

$$\begin{pmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{pmatrix}$$

   $h_{11} = |t_{11}|^2 = |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 \Longrightarrow t_{12} = \cdots = t_{1n} = 0 \cdots$

2. Hermitian matrices and real symmetric matrices

   (1) Hermitian matrices are normal matrices

   A is Hermitian $\overset{def}{\Longleftrightarrow}$ $A^* = A \Longrightarrow A^*A = AA = AA^* \Longleftrightarrow A$ is normal

   $\overset{def}{\Longleftrightarrow}$ $A = U\Lambda U^*$ where Λ is diagonal and U is unitary.

   In $C^{n\times n}$ let $\mathcal{RS}$, $\mathcal{H}$, $\mathcal{N}$ and $\mathcal{D}$ be the collections of all real symmetric matrices, Hermitian matrices, normal matrices and diagonalizable matrices. Then

$$\mathcal{RS} \subset \mathcal{H} \subset \mathcal{N} \subset \mathcal{D} \subset C^{n\times n}.$$

(2) Eigenvalue decomposition
For diagonalizable A, $A = X\Lambda X^{-1}$ is eigenvalue decomposition where Λ is eigenvalue matrix and X is eigenvector matrix. For diagonalizable A select d_i LI eigenvectors from $S_A(\lambda_i) = N(A - \lambda_i I)$ to form X. For normal A select d_i orthonormal eigenvectors from $S_A(\lambda_i)$ to form unitary X.

(3) Eigenvalues of Hermitian A
Eigenvalues of $A = A^*$ are real

Proof If $Ax = \lambda x$ and $x \neq 0$, then $x^*Ax = \lambda x^*x$ and $\lambda = \frac{x^*Ax}{x^*x}$ where $x^*x = \|x\|^2 > 0$. But $\overline{x^*Ax} = (x^*Ax)^* = x^*Ax$. So x^*Ax and consequently λ are both real.

(4) Eigenvectors of Hermitian A
For $A^* = A$, $S_A(\lambda_i) \perp S_A(\lambda_j)$ for $\lambda_i \neq \lambda_j$

Proof $Ax_i = \lambda_i x_i$, $x_i \neq 0$; $Ax_j = \lambda_j x_j$, $x_j \neq 0$; and $\lambda_i \neq \lambda_j$.
$\Longrightarrow x_i^*Ax_j = x_i^*\lambda_j x_j = \lambda_j(x_i^*x_j)$; $x_i^*Ax_j = (Ax_i)^*x_j = (\lambda_i x_i)^*x_j = \lambda_i(x_i^*x_j)$.
So $0 = (\lambda_i - \lambda_j)(x_i^*x_j) \Longrightarrow x_i^*x_j = 0 \Longrightarrow x_i \perp x_j$.

(5) For real symmetric A
$\overline{A} = A = A'$, the eigenvectors can be required to be real. Thus in $A = X\Lambda X'$, X is orthogonal, i.e., $X' = X^{-1}$.

3. Examples

Ex1: Determine if $A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ is normal or diagonalizable. If yes, find correspoding Eigenvalue decompositions. (A is real, but not symmetric).

(i) $A'A = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} = AA'$. So, A is not normal.

(ii) $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^4 + 4 = (\lambda - 2i)(\lambda + 2i) \Longrightarrow \lambda_1 = 2i$ and $\lambda_2 = -2i$.
Because all eigenvalues are simple ones, A is diagonalizable.

(iii) $A - \lambda_1 I \longrightarrow \begin{pmatrix} -2i & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow x_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$; $A - \lambda_2 I \longrightarrow \begin{pmatrix} 2i & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow x_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$
Let $X = \begin{pmatrix} 1 & 1 \\ 2i & -2i \end{pmatrix}$, $\Lambda = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$. Then $A = X\Lambda X^{-1}$.

Ex2: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is real symmetric and hence is normal.
$|A - \lambda I| = 0 \Longrightarrow \lambda_1 = 1$ and $\lambda_2 = -1$;
$A - \lambda_1 I \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Longrightarrow u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $A - \lambda_2 I \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Let $U = (u_1, u_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A = U\Lambda U'$.

Ex3: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Does A have two orthogonal eigenvectors? LI eigenvectors?
$A'A \neq AA'$. So A does not have two orthogonal eigenvectors.
A has eigenvalue $\lambda = 1$ with $r = 2$. But $d = \dim[S_A(1)] = \dim[N(A - I)] = 2 - 1 \neq r$. So A does not have 2 LI eigenvectors.

Ex4: Diagonalizable $A \in C^{n\times n}$ has eigenvalue $\lambda_1, .., \lambda_n$. Find eigenvalues for A^2.
By EVD, $A = X\Lambda X^{-1}$. So $A^2 = X\Lambda^2 X^{-1}$. Thus A^2 have eigenvalues $\lambda_1^2, ..., \lambda_n^2$.