L05 Determinant

1. Determinant

   (1) Definition
   Matrix $A = (a_{ij})_{n\times n}$ has $n!$ products of elements from different rows and columns, $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$, where $j_1...j_n$ is a permutation of $12...n$. Let $f(j_1...j_n)$ be the number of interchanges that convert $j_1..., j_n$ to $12...n$. Then

$$\det(A) = |A| = \sum_{j_1,..,j_n} (-1)^{f(j_1...j_n)} a_{1j_1}\cdots a_{nj_n}.$$

   Ex1: By definition, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$; $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}\cdots a_{nn}$;

$$\text{and } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

   (2) Simple equations: $|A'| = |A|$, $|\overline{A}| = \overline{|A|}$ and $|A^*| = \overline{|A|}$.

   (3) Interchanging two rows/columns changes the sign of determinant
   $A \xrightarrow{(i)\leftrightarrow(j)} B \Longrightarrow |B| = -|A|$.
   Pf: For each term in the summation, $f(j_1, ...j_n)$ changed by 1.
   Ex2: If A has two identical rows (columns), then $|A| = 0$.

   (4) Multiplying one row/column by $\alpha \Longrightarrow$ multiplying the determinant by α.
   $A \xrightarrow{\alpha(i)} B \Longrightarrow |B| = \alpha|A|$.
   Ex3: Suppose $(j) = \alpha(i)$ in A. If $\alpha = 0$, then $|A| = 0$.
   If $\alpha \neq 0$, then $A \xrightarrow{\alpha(i)} B \Longrightarrow |B| = \alpha|A|$. $|A| = \frac{1}{\alpha}|B| = 0$.

   (5) If a row/column is a sum of two vectors,
   $|x+y,\ A_2, .., A_n| = |x,\ A_2, .., A_n| + |y,\ A_2, .., A_n|$.

   (6) For $A \in C^{n\times n}$, $|\alpha A| = \alpha^n\,|A|$.

   (7) Type III elementary operation will not change thedeterminant

   $A \xrightarrow{\alpha(i)+(j)\leftrightarrow(j)} B \Longrightarrow |B| = |A|$.
   Ex4: By elementary row operations,

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & -3 & 3 \\ 2 & 1 & 2 \end{vmatrix} = (-3)\begin{vmatrix} 1 & 0 & 8 \\ 0 & 1 & -1 \\ 0 & 1 & -14 \end{vmatrix} = (-3)\begin{vmatrix} 1 & 0 & 8 \\ 0 & 1 & -1 \\ 0 & 0 & -13 \end{vmatrix} = 39.$$

   (8) For $A \in C^{n\times n}$, A is non-singular $\overset{def}{\Longleftrightarrow}$ A^{-1} exists $\Longleftrightarrow \text{rank}(A) = n \overset{*}{\Longleftrightarrow} \det(A) \neq 0$.
   Proof Type III elementary row operations convert A to upper-triangular matrix U.
   $\text{rank}(A) = \text{rank}(U)$ and $\det(A) = \det(U)$
   $\text{rank}(U) = n \Longleftrightarrow$ All diagonal elements of U are non-zeros $\Longleftrightarrow \det(U) \neq 0$.

2. Elementary matrices

(1) Determinants of elementary matrices
$\det[E((i) \leftrightarrow (j))] = -1$; $\det[E(\alpha(i))] = \alpha$ and $\det[E(\alpha(i) + (j) \to (j))] = 1$.

Proof Show the first one. Let $* : (i) \leftrightarrow (j)$. Then
$I \xrightarrow{*} E(*) \Longleftrightarrow |E(*)| = -|I| = -1.$

(2) Determinant of product of elementary matrix and matrix A
$|E[(i) \leftrightarrow (j)]A| = |E((i) \leftrightarrow (j))|\,|A|$, $|E[\alpha(i)]A = |E[\alpha(i)]|\,|A|$ and
$|E[\alpha(i) + (j) \to (j)]A| = |E[\alpha(i) + (j) \to (i)]|\,|A|$

Proof Show the second one. Let $* : \alpha(i)$. Then
$E(*)A = B \Longleftrightarrow A \xrightarrow{*} B \Longrightarrow |B| = \alpha\,|A| = |E(*)|\,|A|.$

(3) For $A \in C^{n\times n}$, A^{-1} exists $\Longleftrightarrow |A| \neq 0 \Longleftrightarrow \text{rank}(A) = n \Longleftrightarrow A = E_1 \cdots E_k$

Proof We show $\text{rank}(A) = n \Longrightarrow A = E_1 \cdots E_k \Longrightarrow |A| \neq 0$.
If $\text{rank}(A) = n$, then $F_k \cdots F_1 A = I_n$ where $F_1, ..., F_k$ are elementary matrices.
So $A = F_1^{-1} \cdots F_k^{-1} = E_1 \cdots E_k$ where $E_1, ..., E_k$ are elementary matrices.
If $A = E_1 \cdots E_k$, then $|A| = |E_1| \cdots |E_k| \neq 0$. .

Ex5 If $A = E_1 \cdots E_k$, then $|A| = |E_1| \cdots |E_k|$.

(4) $A, B \in C^{n\times n}$, then $|AB| = |A| \cdot |B|$

Pf: If $\text{tank}(A) = n$, then $|AB| = |E_1 \cdots E_k B| = |E_1| \cdots |E_k||B| = |A||B|$
If $\text{rank}(A) < n$, then $\text{rank}(AB) \leq \text{rank}(A) < n$. So $|AB| = 0 = 0|B| = |A||B|$.

Ex6: $|A^{-1}| = \frac{1}{|A|}.|AA^{-1}| = |I| \Longrightarrow |A|\,|A^{-1}| = 1 \Longrightarrow \||A^{-1}| = \frac{1}{|A|}$.

(5) $\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A|\,|B|$ and $\begin{vmatrix} A & 0 \\ D & B \end{vmatrix} = |A|\,|B|$.

Pf: Show the first one. By type III elementary row operations, $A \longrightarrow U_1$, $B \longrightarrow U_2$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} U_1 & C_* \\ 0 & U_2 \end{pmatrix}$ where U_1 and U_2 are upper triangular matrices. Then
$|A| = |U_1|$, $|B| = |U_2|$ and $\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = \begin{vmatrix} U_1 & C_* \\ 0 & U_2 \end{vmatrix} = |U_1|\,|U_2| = |A|\,|B|$

3. Cofactor matrix and adjoint matrix
If $|A| \neq 0$, how to find A^{-1}?

(1) Cofactor matrix and adjoint matrix
Deleting ith row and jth column of $A = (a_{ij})_{n\times n}$ to get M_{ij}. $m_{ij} = |M_{ij}|$ is the minor of a_{ij}. $c_{ij} = (-1)^{i+j} m_{ij}$ is the cofactor of a_{ij}. Matrix $C = (c_{ij})_{n\times n}$ is the cofactor matrix of A, and $A_{\#} = C'$ is the adjoint matrix of A.

(2) Property: $AA_{\#} = |A|\, I_n = A_{\#}A$

Comment: Proof is skipped. There are $2n$ formulas for $|A|$ according to n rows and n columns of cofactor expansion

Ex7: $\begin{vmatrix} 1 & 3 & 5 \\ 0 & -3 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} -3 & 3 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} + 5 \begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} = -9 + 18 + 30 = 39$

(3) $|A| \neq 0 \Longrightarrow A^{-1} = \frac{A_{\#}}{|A|}$.