L03 Bases of $R(A)$ and $N(A)$

1. Bases of $R(A)$ and $N(A)$ for A in reduced row echelon form

   (1) Bases of $\mathcal{R}(A)$ and $\mathcal{N}(A)$
   If $A \in C^{m\times n}$ has rank r, then $\dim[\mathcal{R}(A)] = r$. So A has r LI columns that form a basis for $\mathcal{R}(A)$. Also $\dim[\mathcal{N}(A)] = n - r$. So there are $n-r$ LI vectors in C^n that form a basis for $\mathcal{N}(A)$. How can we find these bases?

   (2) Matrices in reduced row echelon form
   A matrix is in reduced row echelon form if

   (i) the number of leading zeros in rows is increasing

   (ii) the first non-zero element is 1, and is the only non-zero element in the column.

   Ex1: $A = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are in the reduced row echelon form.

   (3) Bases of $\mathcal{R}(A)$ and $\mathcal{N}(A)$ when A is in reduced row echelon form.
   The columns of A associated with (ii) form a basis of $R(A)$.
   Solve equation $Ax = 0$ for variables in x associated with the columns in (ii), using other variables as free ones, one can find a basis for $N(A)$.

   Ex2: For A in Ex1, A_1 and A_3 are LI; $A_2 = 2A_1$ and $A_4 = 5A_1 + 3A_3$.
   So $\text{rank}(A) = 2 = \dim[\mathcal{R}(A)]$ and $[A_1,\ A_3]$ is a basis for $R(A)$.
   With $\dim[\mathcal{N}(A)] = n - \text{rank}(A) = 4 - 2 = 2$ and

$$x \in N(A) \iff Ax = 0 \iff \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - 5x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix} \iff x \in \text{Span}\left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}\right].$$

   $\left[\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}\right]$ is a basis for $\mathcal{N}(A)$.

   (4) Find bases of $\mathcal{R}(A)$ and $\mathcal{N}(A)$ via matrix B
   Suppose $Ax = 0 \iff Bx = 0$. Then
   $x \in \mathcal{N}(A) \iff Ax = 0 \iff Bx = 0 \iff x \in \mathcal{N}(B)$. Thus $\mathcal{N}(A) = \mathcal{N}(B)$ share basis.
   If $B_{i_1}, ..., B_{i_r}$ form a basis for $\mathcal{R}(B)$, then $A_{i_1}, ..., A_{i_r}$ form a basis for $\mathcal{R}(A)$.

   Ex3: Suppose $Ax = 0 \iff Bx = 0$, $B = (B_1,\ B_2,\ B_3)$ and $[B_1,\ B_3]$ is a basis for $\mathcal{R}(B)$.

   $A \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = 0 \Longrightarrow B \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = 0$. So A_1 and A_3 are LI.

   $\exists\, x = \begin{pmatrix} x_1 \\ 1 \\ x_3 \end{pmatrix}$ such that $Bx = 0 \Longrightarrow Ax = 0$. So A_2 is a LC of A_1 and A_3.

   Hence $[A_1,\ A_3]$ is a basis for $\mathcal{R}(A)$.

2. Elementary row operations and elementary matrices

   (1) Elementary row operations
      (i) Interchanging ith row and jth row $(r: (i) \leftrightarrow (j))$
      (ii) Multiplying the ith row by $\alpha \neq 0$ $(r: \alpha(i))$
      (iii) Changing the jth row to the original jth row plus $\alpha \times i$th row $(r: \alpha(i) + (j) \rightarrow (j))$

   (2) Elementary matrices
   Elementary row operations on identity matrices produce elementary matrices.

   Ex4: $I_3 \overset{r:2(1)+(3)\to(3)}{- - - - - - - - >} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = E.$

   (3) Effects of multiplying a matrix from left by an elementary matrix.
   Suppose $I \overset{*}{\longrightarrow} E(*)$. Then $E(*)A = B \Longleftrightarrow A \overset{*}{\longrightarrow} B$.

   Ex5: With E in Example 4, $EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 5 & 5 & 5 & 5 \end{pmatrix}$.

   (4) Properties of elementary matrices
   Elementary matrices are non-singular. If E is an elementary matrix, so are E^{-1} and E'.

   Ex6: With elementary matrix E in Example 4, $EF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = I_3$.

   So $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ is an elementary matrix produced by $-2(1) + (3) \rightarrow (3)$.

3. Find bases of $\mathcal{R}(A)$ and $\mathcal{N}(A)$.

   (1) Convert A by a sequence of elementary operations to B in reduced row echelon form.

   Ex7: $A = \begin{pmatrix} 2 & 2 & 6 & 0 \\ 3 & -3 & -3 & 6 \\ 1 & 0 & 1 & 1 \end{pmatrix} \overset{r:(1)\leftrightarrow(3)}{- - - - \longrightarrow} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & -3 & -3 & 6 \\ 2 & 2 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & 3 & 0 \end{pmatrix}$

   $\overset{r:[-(1)](1)+(i)\to(i),\, i=2,3}{- - - - - - - - \longrightarrow} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$

   (2) With the operations in (1)
   $(E_k \cdots E_1)A = B$ and hence $A = (E_1^{-1} \cdots E_k^{-1})B$. So $Ax = 0 \Longleftrightarrow Bx = 0$.
   Pf $Ax = 0 \Longrightarrow (E_k \cdots E_1)Ax = 0 \Longrightarrow Bx = 0 \Longrightarrow (E_1^{-1} \cdots E_k^{-1})Bx = 0 \Longrightarrow Ax = 0$.

   (3) Find bases of $\mathcal{R}(A)$ and $\mathcal{N}(A)$
      (i) Convert A to its reduced row echelon for B
      (ii) Find bases of $\mathcal{R}(B)$ and $\mathcal{N}(B)$
      (iii) If $B_{i1}, ..., B_{ir}$ for a basis of $\mathcal{R}(B)$, then $A_{i1}, .., A_{ir}$ form a basis of $\mathcal{R}(A)$
      A basis of $\mathcal{N}(B)$ is a basis of $\mathcal{N}(A)$.

   Ex8: For A in Ex7, rank$(A) = 2 = \dim[\mathcal{R}(A)]$. $[A_1, A_2]$ is a basis for $\mathcal{R}(A)$.
   $A_3 = A_1 + 2A_2$ and $A_4 = A_1 - A_2$. $\dim[\mathcal{N}(A)] = 4 - 2 = 2, ...$

L04 Matrix inverses

1. Inverses

   (1) Definitions

   For $A \in C^{m\times n}$, B is a left-inverse of $A \overset{def}{\Longleftrightarrow} BA = I_n$.

   C is a right-inverse of $A \overset{def}{\Longleftrightarrow} AC = I_m$.

   For $A \in C^{n\times n}$ D is inverse of $A \overset{def}{\Longleftrightarrow} AD = I_n = DA$.

   (2) Iff-conditions for left-inverses

   Let A^L be any individual left-inverse or the collection of all left-inverses of A. Then

   (a) $A^L \neq \emptyset$ (b) $\text{rank}(A) = n$ (c) $\text{rank}(AB) = \text{rank}(B)\ \ \forall\, B \in C^{n\times p}$

   are equivalent.

   (a)⇒(b): $n = \text{rank}(I_n) = \text{rank}(A^L A) \leq \text{rank}(A) \leq n$. So $\text{rank}(A) = n$.

   (a)⇐(b): The reduced row echelon form for A is $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$. There exists $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ such that $HA = \begin{pmatrix} H_1 A \\ H_2 A \end{pmatrix} = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$. So $H_1 A = I_n$. Hence $H_1 \in A^L$. Hence $A^L \neq \emptyset$.

   (a)⇒(c): $\text{rank}(AB) \leq \text{rank}(B) = \text{rank}(A^L AB) \leq \text{rank}(AB)$. So $\text{rank}(AB) = \text{rank}(B)$.

   (b)⇐(c): $\text{rank}(A) = \text{rank}(AI_n) = \text{rank}(I_n) = n$.

   Comment: Not all A has L-inverse since not all A has full column rank.

   Ex1: For $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B = (1,\, a) \in A^L$ for all $a \in R$. So left-inverse for A is not unique.

   (3) Iff-conditions for right-inverses

   Let A^R be any individual right-inverse or the collection of all right-inverses of A. Then

   (a) $A^R \neq \emptyset$ (b) $\text{rank}(A) = m$ (c) $\text{rank}(BA) = \text{rank}(B)\ \ \forall\, B \in C^{p\times m}$

   are equivalent.

   Proof. Skipped.

   Comments: A^R may or may not be existent, may or may not be unique.

   (4) Iff-conditions for inverse

   Let A^{-1} be a inverse of A. Then

   (a) A^{-1} exists (b) $\text{rank}(A) = n$

   (c) $\text{rank}(BA) = \text{rank}(B)\ \ \forall\, B$ and $\text{rank}(AD) = \text{rank}(D)\ \ \forall\, D$

   are equivalent.

   Proof. **(a)⟹(b):** A^{-1} exists $\Longrightarrow A^L \neq \emptyset \Rightarrow \text{rank}(A) = n$.

   (b)⟹(c): A has full row rank$\Longrightarrow \text{rank}(BA) = \text{rank}(B)$;

   A has full column rank$\Longrightarrow \text{rank}(AD) = \text{rank}(D)$.

   (c)⟹(a): $\text{rank}(BA) = \text{rank}(B)$ for all $B \Longrightarrow A^R \neq \emptyset$.

   $\text{rank}(AD) = \text{rank}(D)$ for all $D \Longrightarrow A^L \neq \emptyset$.

   $A^L = A^L I_n = A^L A A^R = I_n A^R = A^R$. Let $B = A^L = A^R$.

   Then $AB = I_n = BA$. So B is an inverse of A.

   Ex2: If A^{-1} exists, then it is unique.

   Suppose both B_1 and B_2 are inverse of A. Then $B_1 = B_1 I_n = B_1 A B_2 = I_n B_2 = B_2$.

2. Useful properties

   (1) If A^{-1} exists, then $A^L = A^{-1} = A^R$.

   Proof. Show $A^L = A^{-1}$. $\subset$: $B \in A^L \Longrightarrow BA = I \Longrightarrow BAA^{-1} = IA^{-1} \Longrightarrow B = A^{-1}$.
   $\supset$: $A^{-1}A = I \Longrightarrow A^{-1} \in A^L$.

   (2) If $A \in C^{n\times n}$ and $AB = I_n$, then $B = A^{-1}$; If $A \in C^{n\times n}$ and $BA = I_n$, then $B = A^{-1}$.

   Proof. Show the first one. $n = \text{rank}(I_n) = \text{rank}(AB) \le \text{rank}(A) \le n$.
   So $\text{rank}(A) = n$. Hence A^{-1} exists. $AB = I_n \Longrightarrow B = A^{-1}AB = A^{-1}I = A^{-1}$.

   (3) If $(A|I) \xrightarrow{e.r.} (I|B)$, then $B = A^{-1}$.

   Proof. $(A|I) \xrightarrow{e.r.} (I|B)$ implies that $DA = I$ and $DI = B$. So $B = D = A^{-1}$.

3. Some useful inverses

   (1) Simple rules
   $(A')^{-1} = (A^{-1})'$; $(\overline{A})^{-1} = \overline{(A^{-1})}$; $(A^*)^{-1} = (A^{-1})^*$
   $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ where $\alpha \neq 0$; $(AB)^{-1} = B^{-1}A^{-1}$.
   Proof. Skipped.

   (2) Diagonal blocked matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.

   Proof. $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} AA^{-1} & 0 \\ 0 & BB^{-1} \end{pmatrix} = I.$

   (3) Four-blocked matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix}\begin{pmatrix} A_{11.2}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}$$

   where $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

   Pf: $\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix} = \begin{pmatrix} A_{11.2} & 0 \\ 0 & A_{22} \end{pmatrix}$.

$$\begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix}^{-1}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} A_{11.2}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}.$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix}\begin{pmatrix} A_{11.2}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}$$

Ex3: Let $A_{22.1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22.1}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}.$$

Proof. $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22.1}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} = I$

Ex4: $(A|I) = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{e.r.} \cdots \xrightarrow{e.r.} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$.

So $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$