M344 12 Mar '20

Section 15.9 Transformations of the plane

Calc I. Ex $\int x\,(x^2+3)^9\,dx$

let $u = x^2+3$

$du = 2x\,dx$

$\rightarrow \frac{1}{2x}\,du = dx$

$$= \int \cancel{x}\,(u)^9 \boxed{\frac{1}{2\cancel{x}}\,du}$$

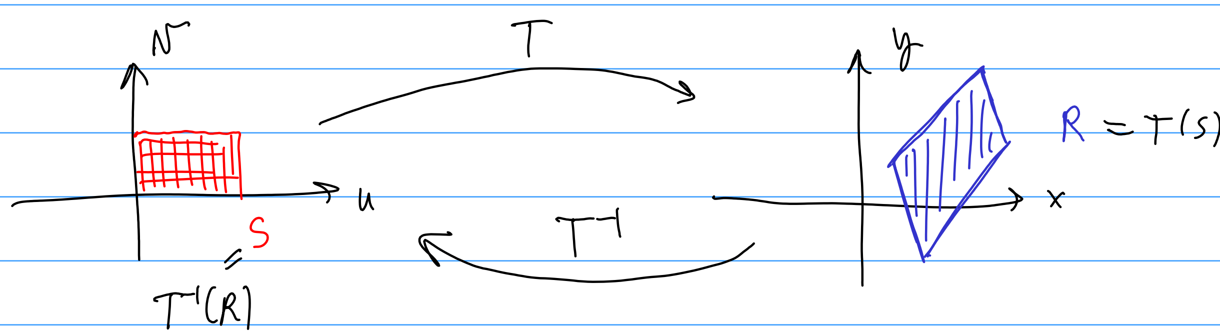
$\frac{du}{dx} = 2x \rightarrow \frac{dx}{du} = \frac{1}{2x}$

$$= \frac{1}{2}\int u^9\,du$$

So $dx = \frac{dx}{du}\,du$

A transformation of the plane is a function

$$T(u,v) = (x,y) = \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

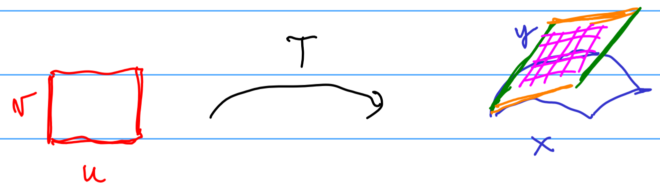


$$\iint_S f(x(u,v),y(u,v))\,dA_{(u,v)} \longrightarrow \iint_R f(x,y)\,dA_{(x,y)}$$

$$= \iint_S f(u,v)\,dA_{(u,v)}$$?

Q. How is $dA_{(x,y)}$ related to $dA_{(u,v)}$?

The transformation T takes rectangles in (u,v) plane to other shapes in the (x,y)-plane



The Jacobian matrix of T is

$$\hat{J}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

The Jacobian of T is the determinant of $\hat{J}$

$$\boxed{J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}} \quad \circledast$$

$$\boxed{dA_{(x,y)} = dx\,dy = J(u,v)\cdot dA_{(u,v)} = J(u,v)\,du\,dv} \quad \circledast$$

Ex. $T(u,v) = \begin{cases} x = u^2 - 2v \\ y = 2uv + v^2 \end{cases}$

$$\hat{J}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 2v \\ -2 & 2u+2v \end{pmatrix}$$

$$J(u,v) = 4u^2 + 4uv - (-4v) = 4(u^2 + uv + v)$$

$$\iint_R f(x,y)\,dA_{(x,y)} = \iint_{T^{-1}(R)} f(u,v)\cdot 4(u^2+uv+v)\,dA_{(u,v)}$$

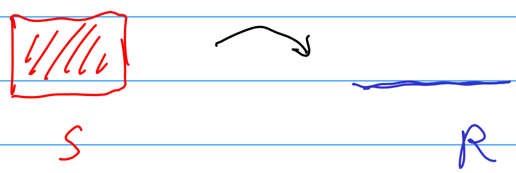
Ex. $T(u,v) = \begin{cases} x = \dfrac{u}{u+v} \\ y = \dfrac{u}{u-v} \end{cases}$ Compute the Jacobian.

$$\frac{\partial x}{\partial u} = \frac{(u+v)-u}{(u+v)^2} = \frac{v}{(u+v)^2} \qquad \frac{\partial y}{\partial u} = \frac{(u-v)-u}{(u-v)^2} = \frac{-v}{(u-v)^2}$$

$$\frac{\partial x}{\partial v} = \frac{-u}{(u+v)^2} \qquad \frac{\partial y}{\partial v} = \frac{-u\cdot(-1)}{(u-v)^2} = \frac{u}{(u-v)^2}$$

$$\hat{J}(u,v) = \begin{pmatrix} v/(u+v)^2 & -v/(u-v)^2 \\ -u/(u+v)^2 & u/(u-v)^2 \end{pmatrix}$$

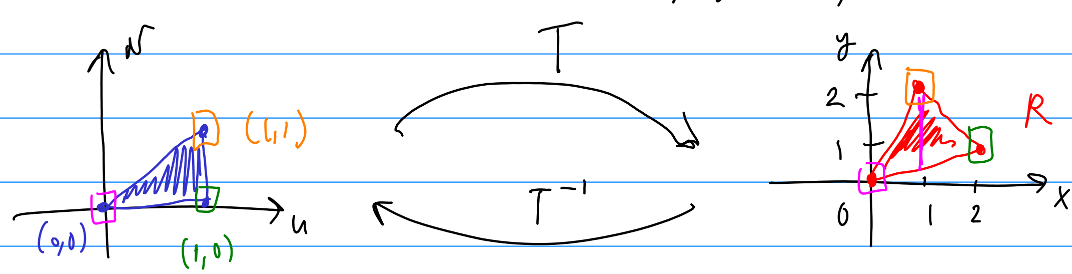
$$J(u,v) = \frac{uv - uv}{(u+v)^2(u-v)^2} = 0$$



$$dA_{(x,y)} = 0\, dA_{(u,v)}$$

Thm. A transformation T is said to be a <u>change-of-coordinates</u> iff the Jacobian of T is not zero.

Ex. $\iint_R (x-3y)\, dA$ where R is the triangle w/ vertices $(0,0)$, $(2,1)$, and $(1,2)$



$$T^{-1}(x,y) = \begin{cases} u = Ax + By \\ v = Cx + Dy \end{cases}$$

D: $T^{-1}(2,1) = (1,0)$

$T^{-1}(1,2) = (1,1)$

$$\begin{cases} 1 = 2A + B \\ 1 = A + 2B \end{cases}$$

$$\begin{cases} 0 = 2C + D \\ 1 = C + 2D \end{cases}$$

$B = 1 - 2A$

$1 = A + 2(1 - 2A)$

$1 = A + 2 - 4A$

$-1 = -3A$

$A = 1/3 \qquad B = 1/3$

$D = -2C$

$1 = C + 2(-2C) \qquad 1 = -3C \qquad C = -1/3$

$D = 2/3$

$$T^{-1}(x,y) = \begin{cases} u = \frac{1}{3}x + 1/3\, y \\ v = \frac{-1}{3}x + \frac{2}{3}y \end{cases}$$

To invert this, solve for x, y.

$u + v = y$

$u = \frac{1}{3}x + \frac{1}{3}(u+v)$

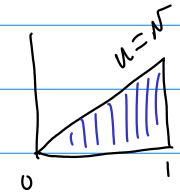
$3u = x + u + v$

$x = 2u - v$

$$T(u,v) = \begin{cases} x = 2u - v \\ y = u + v \end{cases}$$

$$\iint_R (x - 3y)\, dA = \iint_S \big(2u - v - 3(u+v)\big)\, J(u,v)\, dA_{(u,v)}$$

S:

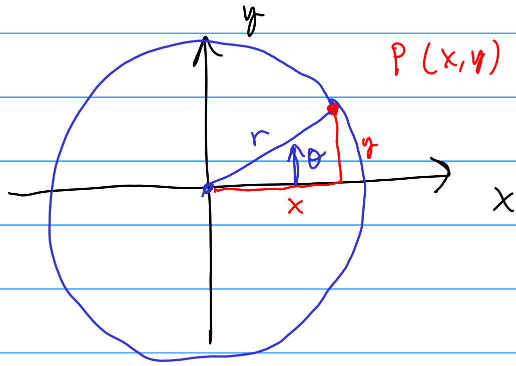


$$\hat{J}(u,v) = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{so} \quad J(u,v) = 2 + 1 = 3$$

$$\iint_R f(x,y)\, dA = \int_0^1 \int_0^v (-u - 4v)\, 3\, du\, dv = \text{DO IT!}$$

-3 ?

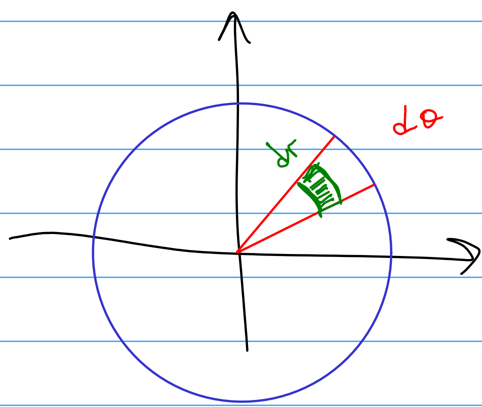
15.3 - Polar Coordinates!



$$T(r,\theta) = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$T^{-1}(x,y) = \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x)_{\tilde{Q}} \end{cases}$$

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } \theta \in QI, QIV, \text{ so if } x>0 \\ \tan^{-1}(y/x)+\pi & \text{if } x<0 \\ \pi/2 & \text{if } x=0,\ y>0 \\ 3\pi/2 & \text{if } x=0,\ y<0 \end{cases}$$

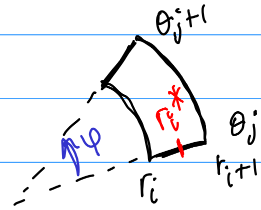


$$dA = dx\,dy = J(r,\theta)\,dr\,d\theta = r\,dr\,d\theta$$

1. Jacobian: $T(r,\theta) = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \hat{J} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$

$$J(r,\theta) = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

2. Use Geometry:



$$d\theta_j = \theta_{j+1} - \theta_j$$
$$dr_i = r_{i+1} - r_i$$
$$r_i^* = \frac{1}{2}(r_{i+1} + r_i)$$

Area of a sector is $\frac{1}{2} r^2 \cancel{\theta}\varphi$

The area $dA_{ij} = \frac{1}{2} r_{i+1}^2 \, d\theta_j - \frac{1}{2} r_i^2 \, d\theta_j$

$$= \frac{1}{2}\left(r_{i+1}^2 - r_i^2\right) d\theta_j$$

$$= \underbrace{\frac{1}{2}\left(r_{i+1} + r_i\right)}_{r_i^*} \underbrace{\left(r_{i+1} - r_i\right)}_{dr_i} d\theta_j$$

$$dA_{ij} = r_i^* \, dr_i \, d\theta_j \longrightarrow dA_{(r,\theta)} = r \, dr \, d\theta.$$