M344 10 March 2020

## Riemann Sum!

$$\iint_R x+y^2 \, dA \qquad R=[0,1]\times[0,2]$$

$$\sum_{j=1}^{m} 1 = m$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \underline{\lim_{m\to\infty} \sum_{j=1}^{m}} \left(x_i^* + (y_j^*)^2\right) \Delta x_i \Delta y_j$$

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$$

$$\Delta y_j = \frac{b-a}{m} = \frac{2-0}{m} = \frac{2}{m}$$

$$y_j^* = a + j\Delta y = 0 + j\frac{2}{m} = \frac{2j}{m}$$

$$\sum_{j=1}^{m} j^2 = \frac{m(2m+1)(m+1)}{6}$$

$$(y_j^*)^2 = \frac{4}{m^2} j^2$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left[\lim_{m\to\infty} \sum_{j=1}^{m} \left(x_i^* + \frac{4}{m^2} j^2\right) \frac{2}{m}\right] \Delta x_i$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left[\lim_{m\to\infty} \left(\frac{2x_i^*}{m} \sum_{j=1}^{m} 1 + \frac{8}{m^3} \sum_{j=1}^{m} j^2\right)\right] \Delta x_i$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left[\lim_{m\to\infty} \left(\frac{2x_i^* m}{m} + \frac{8}{m^3} \frac{m(m+1)(2m+1)}{6}\right)\right] \Delta x_i$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left(2x_i^* + \frac{8}{3}\right) \Delta x_i$$

$$\Delta x_i = \Delta x = \frac{b-a}{N} = \frac{1-0}{N} = \frac{1}{N}$$

$$x_i^* = a + i\Delta x = 0 + \frac{i}{N} = \frac{i}{N}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left(2\frac{i}{N} + \frac{8}{3}\right) \frac{1}{N} = \lim_{N\to\infty} \left(\frac{2}{N^2} \sum_{i=1}^{N} i + \frac{8}{3N} \sum_{i=1}^{N} 1\right)$$

$$= \lim_{N\to\infty} \left(\frac{2}{N^2} \frac{N(N+1)}{2} + \frac{8N}{3N}\right) = 1 + \frac{8}{3} = \boxed{\frac{11}{3}}$$

The FTC applies to each variable individually, so we get Fubini's Theorem:

Let $f = f(x,y)$ be continuous on $R = [a,b] \times [c,d]$. Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex. $\displaystyle\iint_R y\sin(xy)\,dA \qquad R = [1,2] \times [0,\pi]$

$$\int_0^{\pi} \left[ \int_1^2 y\sin(xy)\,dx \right] dy = \int_0^{\pi} \left[ -\frac{\cancel{y}}{\cancel{y}}\cos(xy) \Big|_{x=1}^{2} \right] dy$$

$$= \int_0^{\pi} \left( -\cos(2y) + \cos(y) \right) dy$$

$$= -\frac{1}{2}\sin(2y) + \sin y \Big|_0^{\pi} = 0.$$

Try the other direction.

$$\int_1^2 \left[ \int_0^{\pi} y\sin(xy)\,dy \right] dx = \int_1^2 \left[ \frac{-y}{x}\cos(xy) \Big|_{0=y}^{\pi} + \frac{1}{x}\int_0^{\pi} \cos(xy)\,dy \right] dx$$

$u = y \qquad dv = \sin(xy)\,dy$

$du = dy \qquad v = \frac{-1}{x}\cos(xy)$

$$= \int_1^2 \left[ \frac{-y}{x}\cos(xy) \Big|_{y=0}^{\pi} + \frac{1}{x^2}\sin(xy) \Big|_{y=0}^{\pi} \right] dx$$

$$= \int_1^2 \frac{-\pi}{x}\cos(\pi x) + \frac{1}{x^2}\sin(\pi x)\,dx$$

Use Power Series!

$$\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} u^{2n} \qquad \sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u^{2n+1}$$

$$\cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2n!} x^{2n} \qquad \sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$-\frac{\pi}{x} \cos(\pi x) = -\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n)!} x^{2n-1}$$

$$\frac{1}{x^2} \sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n-1}$$

$$-\frac{\pi}{x} \cos(\pi x) + \frac{1}{x^2} \sin(\pi x) = \sum_{n=0}^{\infty} (-1)^n \pi^{2n+1} \left( \frac{1}{(2n+1)!} - \frac{1}{(2n)!} \right) x^{2n-1}$$

$$= \overset{0}{\cancel{\pi \left( \frac{1}{1!} - \frac{1}{0!} \right) \frac{1}{x}}} + \sum_{n=1}^{\infty} (-1)^n \pi^{2n+1} \left( \frac{1}{(2n+1)!} - \frac{1}{(2n)!} \right) x^{2n-1}$$

So $\displaystyle\int_1^2 \sum_{n=1}^{\infty} (-1)^n \pi^{2n+1} \left( \frac{1}{(2n+1)!} - \frac{1}{(2n)!} \right) x^{2n-1} \, dx$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2n} \left( \frac{1}{(2n+1)!} - \frac{1}{(2n)!} \right) x^{2n} \Bigg|_1^2 = \text{More work.}$$

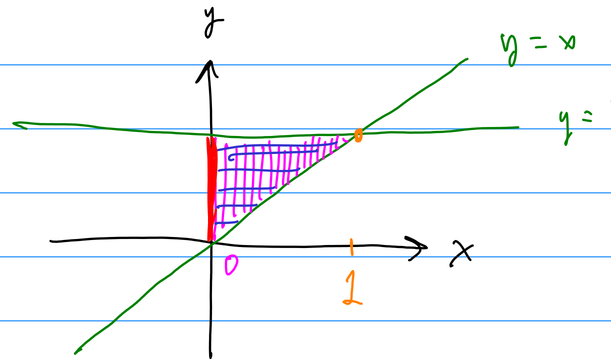
$(2n+1)! = (2n+1)(2n)!$ so

$$\frac{1}{(2n+1)!} - \frac{2n+1}{(2n+1)!} = \frac{1-2n-1}{(2n+1)!} = \frac{-2n}{(2n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{(2n+1)!} x^{2n} \Bigg|_1^2$$

**Ex.** $\displaystyle\int_0^1 \left[\int_x^1 \sin(y^2)\,dy\right]dx$

$\underbrace{\int_0^1\int_x^1}_{\text{Not a rectangle!}}$




$y: \; y = x \to y = 1$

$x: \; x = 0 \longrightarrow x = 1$

Switch: $x = 0 \to x = y$

$y = 0 \to y = 1$

$$\int_0^1\int_x^1 \sin(y^2)\,dy\,dx = \int_0^1\left[\int_0^y \sin(y^2)\,dx\right]dy$$

$$= \int_0^1 \left[\; x\sin(y^2)\Big|_{x=0}^{y} \;\right] dy$$

$$= \frac{1}{2}\int_0^1 2y\sin(y^2)\,dy \qquad u = y^2 \qquad u(1) = 1$$

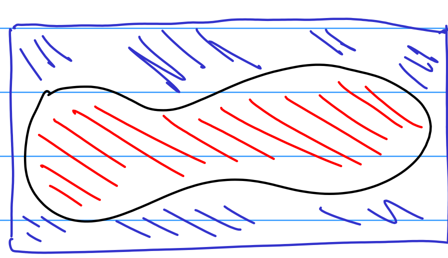
$$du = 2y\,dy \qquad u(0) = 0$$

$$= \frac{1}{2}\int_0^1 \sin(u)\,du = -\frac{1}{2}\cos u\Big|_0^1$$

$$= -\frac{1}{2}\cos(1) + \frac{1}{2}$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}$$

## 15.2 Integrals over general regions.

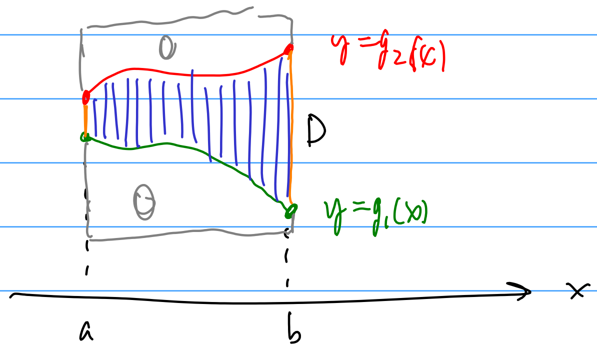


D

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

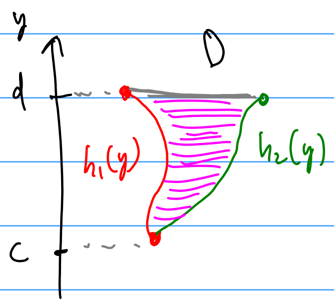
$$F(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin D \\ f(x,y) & \text{if } (x,y) \in D \end{cases}$$

Type I: $D$ is bounded between $y = g_1(x)$ and $y = g_2(x)$ between $a \le x \le b$.




$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

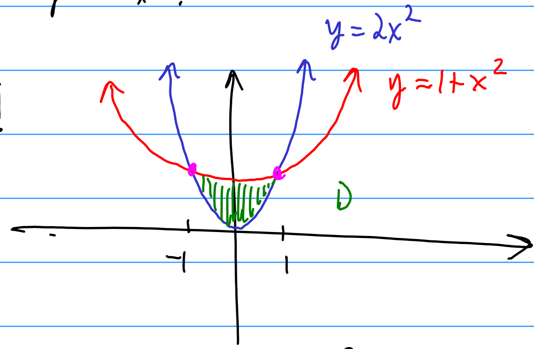
Type II: $x = h_1(y)$ and $x = h_2(y)$ between $c \le y \le d$.




$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Ex. $\iint_D x + 2y \; dA$    $D$ is the region bounded between $y = 2x^2$ and $y = 1 + x^2$.

Type I!




$$2x^2 = 1 + x^2$$
$$x^2 = 1$$
$$x = \pm 1$$

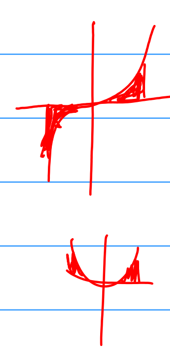
$$\iint_D x + 2y\, dA =$$

$$= \int_{-1}^{1} \left[ \int_{2x^2}^{1+x^2} x + 2y \; dy \right] dx$$

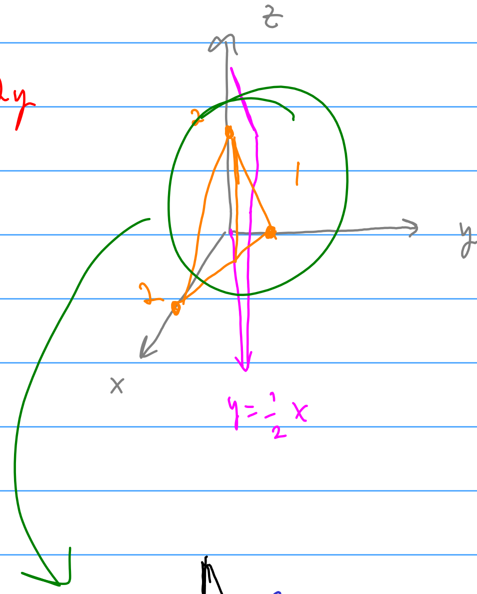
$$= \int_{-1}^{1} \left[ xy + y^2 \Big|_{y=2x^2}^{1+x^2} \right] dx$$

$$= \int_{-1}^{1} \left[ \cancel{x(1+x^2) - x(2x^2)} + (1+x^2)^2 - (2x^2)^2 \right] dx$$

$$= 2 \int_0^1 1 + 2x^2 + x^4 - 4x^4 \; dx = 2\left(1 + \tfrac{2}{3} - \tfrac{3}{5}\right) = \boxed{\tfrac{32}{5}}$$



Ex. Find the volume of the tetrahedron bdd by the planes

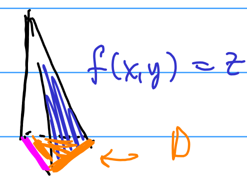



$\boxed{x+2y+z=2}$ ← $f(x,y)=2-x-2y$

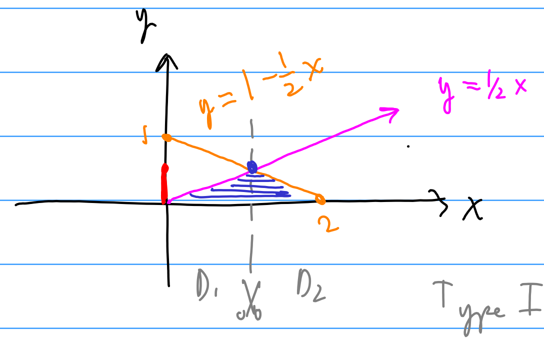
$x=2y$

$x=0$

$z=0$




volume $= \iint\limits_D f(x,y)\,dA$




Type II

$x:\ 2-2y \longrightarrow 2y$

$y:\ 0 \longrightarrow 1/2$

$$V = \int_0^{1/2}\int_{2-2y}^{2y}(2-x-2y)\,dx\,dy$$

Integrate!