M344 27 Feb 20

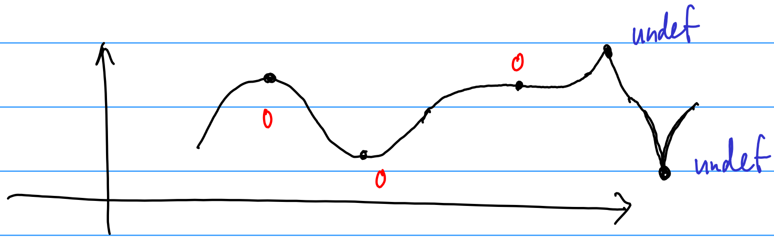
14.7–8 – Optimization

Defn. A critical point of a function f of multiple variables is a point P in the domain of f such that

$$\nabla f(P) = \vec{0}$$
$$\text{or} \quad \nabla f(P) = \text{undef.}$$

Recall from Calc I,

Fermat's Theorem. If f has a local max or min at p, then p is a critical number of f.



Theorem. If $f = f(x,y)$ has a local max or min at $P(x_0, y_0)$, then P is a critical point of f.

Recall, the Hessian matrix of f is

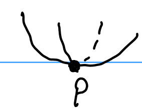
$$\hat{H}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

The Hessian function is $H(f) = \det(\hat{H}(f)) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

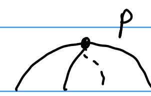
Second Derivative Test

Let f be a function of two variables with continuous 2^{nd} partials, and suppose P is a critical point of f, so $\nabla f(P) = \vec{0}$.

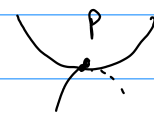
Case 1.) If $H_f(P) > 0$ and if $\frac{\partial^2 f}{\partial x^2}(P) > 0$ then $(P, f(P))$ is a local minimum.



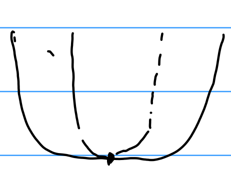
Case 2.) If $H_f(P) > 0$ and if $\frac{\partial^2 f}{\partial x^2}(P) < 0$ then $(P, f(P))$ is a local maximum.

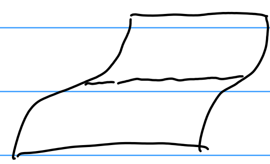


Case 3.) If $H_f(P) < 0$, then $(P, f(P))$ is a saddle point.



If $H_f(P) = 0$, the second derivative test does not apply.





Ex. $f(x,y) = x^2 - 2xy + 2y^2$

Find all critical points, then classify them.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle \underbrace{2x - 2y}_{\partial x}, \underbrace{-2x + 4y}_{\partial y} \rangle = \langle 0, 0 \rangle$$

$$\begin{array}{rl} & 2x - 2y = 0 \\ + & -2x + 4y = 0 \\ \hline & 2y = 0 \end{array} \rightarrow \begin{array}{l} y = 0 \\ x = 0 \end{array}$$

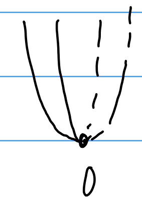
$(0,0)$ is the only CP.

$$\hat{H}_f = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \qquad \det(\hat{H}_f) = H_f = 8-(4) = 4 > 0$$

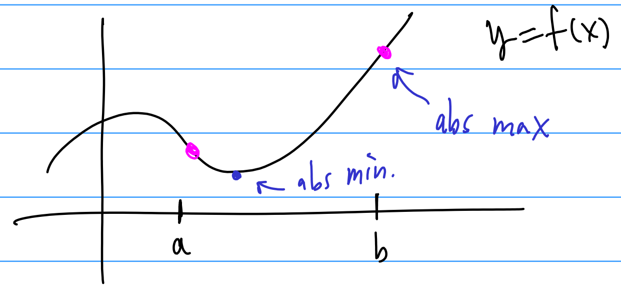
for all points.

$H_f(P) > 0$. $\quad \frac{\partial^2 f}{\partial x^2}(P) = 2 > 0$.

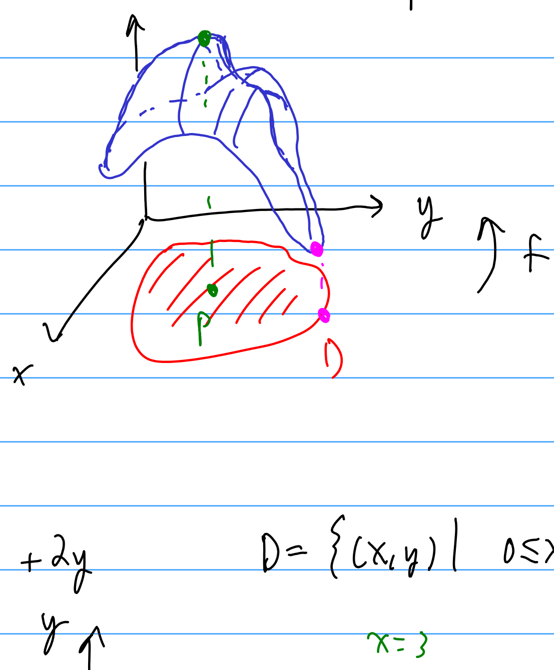
So P is a local minimum.



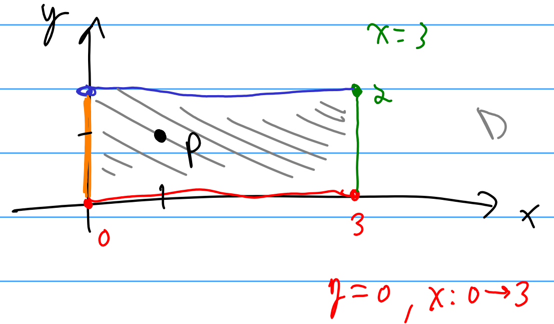
<u>Extreme Value Theorem</u>



Let f be a function of two variables that is continuous on a closed region D.
Then f attains an absolute max and min on D.
These extreme value either occur at critical points or on the boundary.



Ex. $f(x,y) = x^2 - 2xy + 2y \qquad D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$



$f(x,y) = x^2 - 2xy + 2y$

$\nabla f = \langle 2x - 2y, -2x + 2 \rangle = \langle 0, 0 \rangle$

$\begin{cases} 2x - 2y = 0 \\ -2x + 2 = 0 \end{cases} \leftarrow$ $x = 1$, $y = 1$

$CP = (1,1)$

$f(P) = f(1,1) = 1 - 2 + 2 = 1$

set $y = 0$, $0 \le x \le 3$.

$f(x,0) = x^2$ use Calc I on this!

$f'(x) = 2x$ $x = 0$ already on the boundary.

$f(0,0) = 0^2 = 0$

$f(3,0) = 3^2 = 9$

$x = 3$ $0 \le y \le 2$: $f(3,y) = 9 - 6y + 2y = 9 - 4y$

$f(3,0) = 9$

$f(3,2) = 9 - 8 = 1$

$y = 2$ $0 \le x \le 3$: $f(x,2) = x^2 - 4x + 4 = (x-2)^2$ $x = 2$ is CP.

$f(3,2) = 1$

$f(2,2) = 0$

$f(0,2) = 4$

$x = 0$, $0 \le y \le 2$, $f(0,y) = 2y$

$f(0,0) = 0$

$f(0,2) = 4$

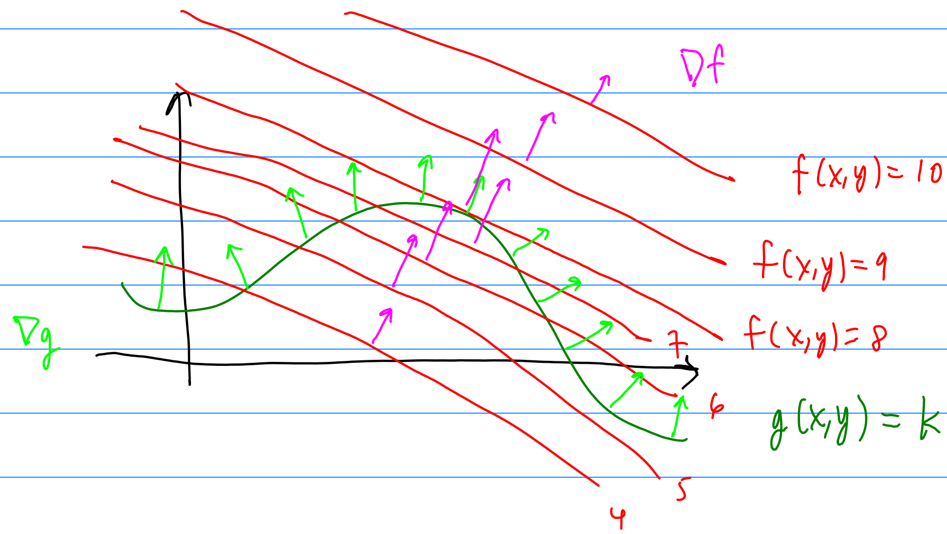
Absolute max is 9 at $(3,0)$

Absolute min is 0 at $(0,0)$ and $(2,2)$

Lagrange Multipliers

Let $f(x,y)$ be a function of two variable, differentiable.
(∇f exists at all points).
Let $g(x,y)$ be another differentiable function.

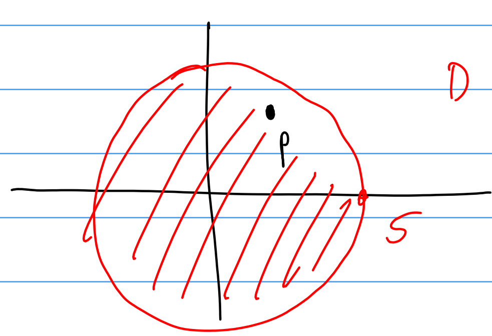
Suppose we wish to optimize f subject to the constraint $g(x,y)=k$.



At the points of optimization $\nabla f(P) = \lambda \nabla g(P)$ for some $\lambda \in \mathbb{R}$.

We get a system of 3 equations and 3 unknowns $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=k. \end{cases}$

Ex. $f(x,y) = x^2-4x+y^2-9y$ $\quad D = \quad x^2+y^2 \le 25$



$\nabla f = \langle 2x-4, 2y-9 \rangle = \langle 0,0 \rangle$ $\quad \begin{cases} 2x-4=0 \\ 2y-9=0 \end{cases} \rightarrow \left.\begin{matrix} x=2 \\ y=9/2 \end{matrix}\right\} P$

$$f(2, 9/2) = 2^2 - 4(2) + (9/2)^2 - 9(9/2) = -4 - \frac{81}{4} = -\frac{97}{4} \approx -24.25$$

Two ways to handle the boundary: $x^2+y^2=5^2$

1. Parametrize it and use calc I. $\begin{cases} x = 5\cos t \\ y = 5\sin t \end{cases}$ $0 \le t \le 2\pi$

$$f(x,y) = f(5\cos t, 5\sin t) = f(t).$$

2. Use Lagrange: $g(x,y) = x^2+y^2$ w/ restriction $g(x,y)=25$

$$\nabla g = \langle 2x, 2y \rangle$$

$\nabla f = \lambda \nabla g$: $\langle 2x-4, 2y-9 \rangle = \lambda \langle 2x, 2y \rangle$

$$\begin{cases} 2x-4 = 2\lambda x \\ 2y-9 = 2\lambda y \\ x^2+y^2 = 25 \end{cases}$$

$2x - 2\lambda x = 4$ $\qquad$ $2y - 2\lambda y = 9$

$x(2-2\lambda) = 4$ $\qquad$ $y(2-2\lambda) = 9$

$x = \dfrac{4}{2-2\lambda}$ $\qquad$ $y = \dfrac{9}{2-2\lambda}$

$$\left(\frac{4}{2-2\lambda}\right)^2 + \left(\frac{9}{2-2\lambda}\right)^2 = 25$$

$\frac{1}{4} \frac{97}{(1-\lambda)^2} = 25 \to (1-\lambda)^2 = \frac{97}{100} \Rightarrow 1-\lambda = \pm\frac{\sqrt{97}}{10}$

$(\lambda-1)^2 = \frac{97}{100}$

$$\lambda = 1 \pm \frac{\sqrt{97}}{10}$$

$$x = \frac{2}{1-\lambda} \qquad x = \frac{2}{\frac{\pm\sqrt{97}}{10}} = \frac{\pm 20}{\sqrt{97}}$$

$$y = \frac{9/2}{1-\lambda} = \frac{9/2}{\frac{\pm\sqrt{97}}{10}} = \pm\frac{45}{\sqrt{97}}$$

Boundary pts:

$$\left(\frac{20}{\sqrt{97}}, \frac{45}{\sqrt{97}}\right)$$

$$\left(\frac{-20}{\sqrt{97}}, \frac{-45}{\sqrt{97}}\right)$$

$$f\left(\frac{20}{\sqrt{97}}, \frac{45}{\sqrt{97}}\right) = \left(\frac{20}{\sqrt{97}}\right)^2 - 4\left(\frac{20}{\sqrt{97}}\right) + \left(\frac{45}{\sqrt{97}}\right)^2 - 9\left(\frac{45}{\sqrt{97}}\right)$$

$$= \frac{400+2025}{97} - \frac{80+405}{\sqrt{97}} = \frac{2425-485\sqrt{97}}{97}$$

$$\approx \cancel{-24.24}$$

$$f\left(\frac{-20}{\sqrt{97}}, \frac{-45}{\sqrt{97}}\right) = \frac{2425+485\sqrt{97}}{97}$$ MAX

-24.25 MIN