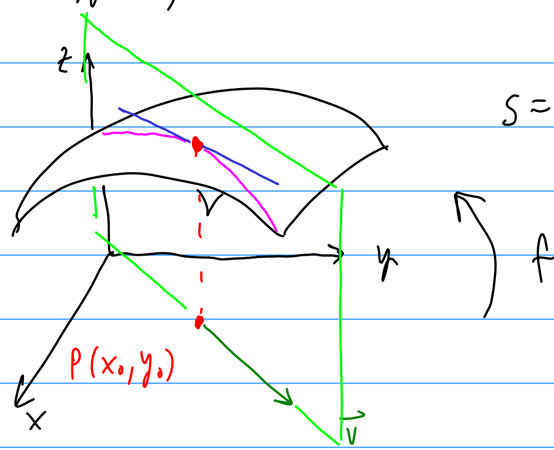
M344 2/25/2020

14.6 Directional Derivatives

$z = f(x,y)$ at (x_0, y_0), and a vector $\vec{v} = \langle A, B \rangle$.



1. Only the direction of $\vec{v}$ should contribute to the directional derivative.
   Use $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \langle a, b \rangle$.

2. The directional derivative of f at (x_0, y_0) in the direction of $\langle a, b \rangle$ is the slope of the tangent line in the picture.

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}} \qquad (*)$$

Special Cases. 1. $\vec{u} = \langle 1, 0 \rangle$

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \frac{\partial f}{\partial x} = D_x f$$

2. $\vec{u} = \langle 0, 1 \rangle \longrightarrow D_{\vec{u}} f = \dfrac{\partial f}{\partial y} = D_y f$

Ex. $f(x,y) = x + y^2$ $\quad \vec{v} = \langle 1,1 \rangle \rightsquigarrow \vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$D_{\vec{v}} f = D_{\vec{u}} f = \lim_{h \to 0} \frac{(x + \frac{1}{\sqrt{2}}h) + (y + \frac{1}{\sqrt{2}}h)^2 - (x + y^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \frac{1}{\sqrt{2}}\cancel{h} - \cancel{x}}{\cancel{h}} + \lim_{h \to 0} \frac{\cancel{y^2} + \sqrt{2}y\cancel{h} + \frac{1}{2}h^{\cancel{2}} - \cancel{y^2}}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(\frac{1}{\sqrt{2}} + \sqrt{2}y + \frac{1}{2}h \right) = \frac{1}{\sqrt{2}} + \sqrt{2}y$$

How do we actually compute this? $\quad D_{\vec{u}} f(x_0, y_0)$, $P(x_0, y_0)$, $\vec{u} = \langle a, b \rangle$
$\|\vec{u}\| = 1$

Consider $g(h) = f(x_0 + ah, y_0 + bh)$

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h - 0} = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

On the other hand, $g(h) = f(x(h), y(h))$

where $\begin{cases} x = x_0 + ah \\ y = y_0 + bh \end{cases}$

$$\frac{dg}{dh} = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}$$

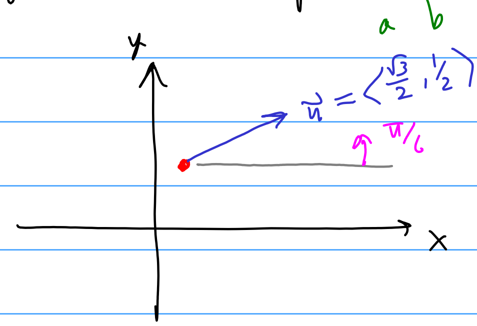
and $\quad g'(0) = a \frac{\partial f}{\partial x}(x_0, y_0) + b \frac{\partial f}{\partial y}(x_0, y_0)$

Theorem. The directional derivative of f in the direction $\vec{u} = \langle a, b \rangle$, $\|\vec{u}\| = 1$, is

$$\boxed{D_{\vec{u}} f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}} \qquad (**)$$

Ex. $f(x,y) = x^3 - 3xy + 4y^2$

Find $D_{\vec{u}} f(1,2)$ where $\vec{u}$ makes an angle $\pi/6$ w/ the positive x-direction.



$$\frac{\partial f}{\partial x} = 3x^2 - 3y \qquad \frac{\partial f}{\partial y} = -3x + 8y$$

$$\frac{\partial f}{\partial x}(1,2) = 3 - 6 = -3 \qquad \frac{\partial f}{\partial y}(1,2) = -3 + 16 = 13$$

$$D_{\vec{u}} f(1,2) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = \frac{\sqrt{3}}{2}(-3) + \frac{1}{2}(13) = \frac{13 - 3\sqrt{3}}{2}$$

Ex. $f(x,y) = x^3 - 3xy + 4y^2 \qquad \vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$$D_{\vec{u}} f(x,y) = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y)$$

$$= \frac{3\sqrt{3}}{2}x^2 - \frac{3\sqrt{3}}{2}y - \frac{3}{2}x + 4y$$

Take $D^2_{\vec{u}} f = D_{\vec{u}}(D_{\vec{u}} f) = a\frac{\partial}{\partial x}(D_{\vec{u}} f) + b\frac{\partial}{\partial y}(D_{\vec{u}} f)$

$$\frac{\partial}{\partial x}(D_{\vec{u}} f) = 3\sqrt{3}x - \frac{3}{2} \qquad \frac{\partial}{\partial y}(D_{\vec{u}} f) = \frac{-3\sqrt{3}}{2} + 4$$

$$D^2_{\vec{u}} f = \frac{\sqrt{3}}{2}\left(3\sqrt{3}x - \frac{3}{2}\right) + \frac{1}{2}\left(-\frac{3\sqrt{3}}{2} + 4\right)$$

$$= \frac{9}{2}x - \frac{3\sqrt{3}}{4} - \frac{3\sqrt{3}}{4} + 2 = \boxed{\frac{9}{2}x + \frac{4 - 3\sqrt{3}}{2}}$$

Look at this:

$$D_{\vec{u}}f = a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} \qquad \vec{u} = \langle a, b \rangle$$

$$\boxed{D_{\vec{u}}f = \nabla f \cdot \vec{u}} \quad (3*) \qquad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Definition. A function f is differentiable if all partial derivatives exist.

The gradient vector of f at (x_0, y_0) is the vector given by

$$\text{grad}(f) = \nabla f(x_0, y_0) = \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle$$

Ex. $f(x,y) = e^{2x}\sin(y)$ $\qquad \vec{v} = \langle 3, 4 \rangle$ $\qquad$ Compute $D_{\vec{v}}f$

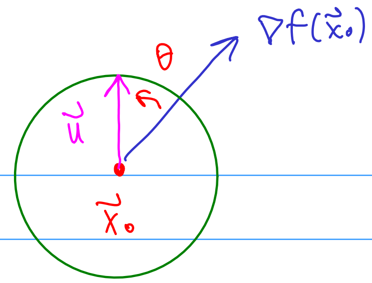
$$\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \longrightarrow \vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\nabla f = \langle 2e^{2x}\sin y, e^{2x}\cos y \rangle$$

$$D_{\vec{v}}f = \nabla f \cdot \vec{u} = \frac{6}{5}e^{2x}\sin(y) + \frac{4}{5}e^{2x}\cos(y).$$

Geometric Significance of Gradient.

Theorem. Let f be a function of at least two variables. Then the maximum value of the directional derivative of f at $\vec{x}_0$ is $\|\nabla f(\vec{x}_0)\|$ and it occurs when $\vec{u}$ is in the same direction as $\nabla f(\vec{x}_0)$.



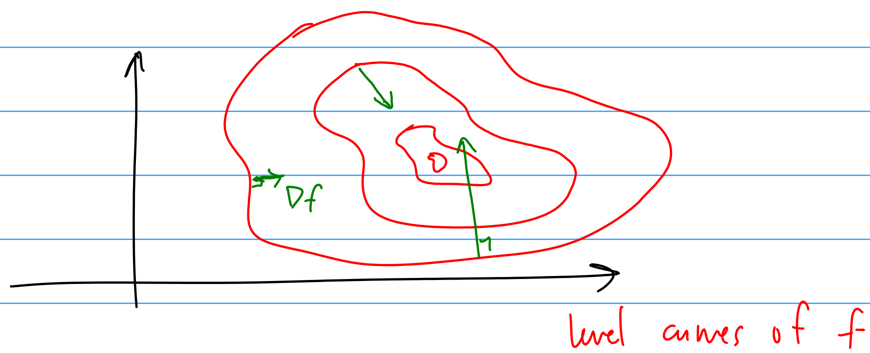
$\vec{u}$ varies.

<u>Proof.</u> The directional derivative $D_{\vec{u}} f(\vec{x}_0)$ is given by

$$
\begin{aligned}
D_{\vec{u}} f(\vec{x}_0) &= \nabla f(\vec{x}_0) \cdot \vec{u} \\
&= \|\nabla f(\vec{x}_0)\| \, \|\vec{u}\| \cos\theta \\
&= \|\nabla f(\vec{x}_0)\| \cos\theta
\end{aligned}
$$

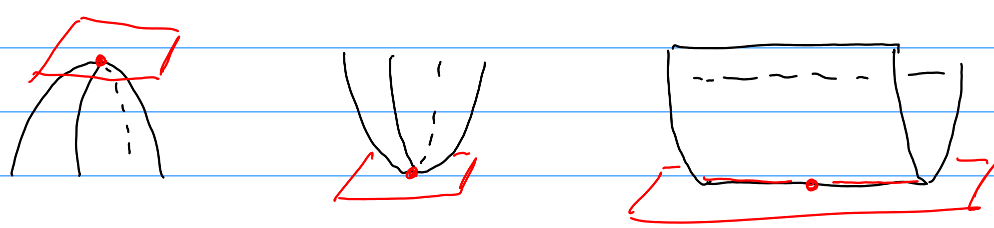
and maximum value of $\cos\theta = 1$, whence $\max(D_{\vec{u}} f) = \|\nabla f(\vec{x}_0)\|$.
this occurs when $\theta = 0$, and therefore $\vec{u}$ and $\nabla f(\vec{x}_0)$ have the same direction. $\blacksquare$

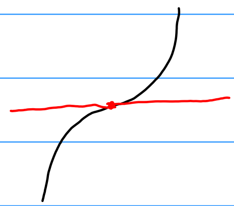
Geometrically, this tells us that the gradient <u>always</u> point in the direction of steepest ascent (uphill).

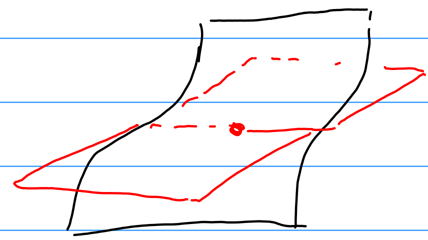


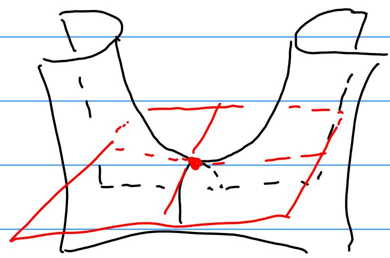
There is a possibility that $\nabla f = \vec{0}$. How?

<u>Case 1.</u>

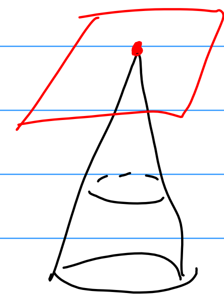








$Df = $ undef.



Def'n. Let f be a function with 2^{nd} partial derivatives.

The Hessian matrix of f is

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

The Hessian function of f is $\hat{H}(f) = \det(H(f))$

$$\hat{H}(f) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$ provided Clairaut applies.

The Laplacian of f is $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

or $\Delta f = tr\,(H(f))$.