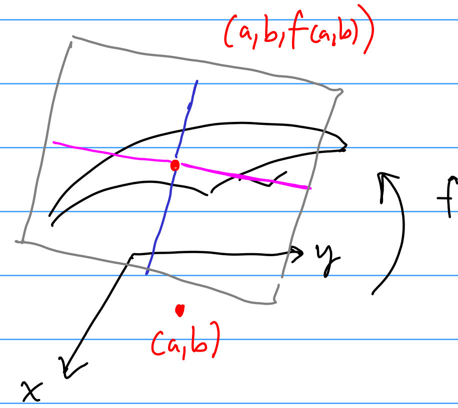
M344 — 18 Feb 2020

§14.4 Differentials

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

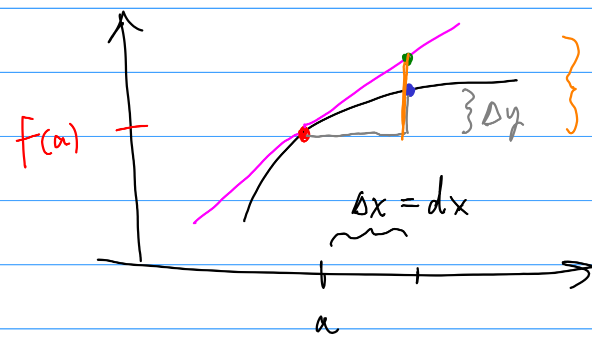


$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Tangent Plane

As a function: Linear Approximation.

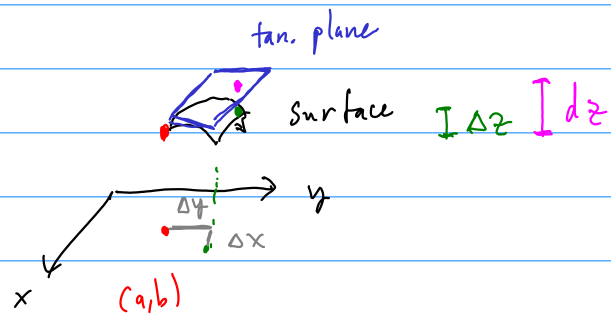
Calc I: $y = f(x)$



$dy = f'(a)\,dx$

differential of y.

Calc III: $z = f(x,y)$



$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$dz = \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y = \boxed{dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$

Total Differential

Now, $f(a+dx, y+dy) \approx f(a,b) + dz$ Linear Approx w/ differentials.

Ex. $f(x,y) = x^2 y \sin(x)$

$$\frac{\partial f}{\partial x} = 2xy\sin(x) + x^2 y\cos(x)$$

$$\frac{\partial f}{\partial y} = x^2 \sin(x)$$

$$dz = (2xy\sin(x) + x^2 y\cos(x))\,dx + x^2\sin(x)\,dy$$

Use this to approximate $f(\frac{\pi}{2}-0.1, 2.2)$

Use $(a,b) = (\frac{\pi}{2}, 2)$

$f(\frac{\pi}{2}, 2) = \frac{\pi^2}{4} \cdot 2 \cdot \sin(\pi/2) = \frac{\pi^2}{2}$

$dx = -0.1$

$dy = 0.2$

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, 2\right) = 2 \cdot \frac{\pi}{2} \cdot 2\sin(\pi/2) + \frac{\pi^2}{4} \cdot 2 \cdot \cos(\pi/2) = 2\pi$$

$$\frac{\partial f}{\partial y}(\pi/2, 2) = \frac{\pi^2}{4} \cdot \sin(\pi/2) = \pi^2/4$$

$$dz = \frac{\partial f}{\partial x}(\pi/2, 2)\,dx + \frac{\partial f}{\partial y}(\pi/2, 2)\,dy = 2\pi(-0.1) + \left(\frac{\pi^2}{4}\right)(0.2) = \left(\frac{\pi^2}{2} - 2\pi\right)0.1 = \frac{\pi^2 - 4\pi}{20}$$

$$f\left(\frac{\pi}{2} - 0.1, 2.2\right) \approx \frac{\pi^2}{2} + \frac{\pi^2 - 4\pi}{20} = \frac{11\pi^2 - 4\pi}{20}$$

14.5 Chain Rule

Case I. $f(x,y) = z$ and $y = y(x)$

then $z = f(x,y) = f(x, y(x)) = f(x)$

Then $\frac{df}{dx}$ should make sense.

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}} \quad (*)$$

Ex. $f(x,y) = e^{x^2}\sin(2y)$ $\qquad y = \frac{x}{2\pi}$

$$f(x) = f\left(x, \frac{x}{2\pi}\right) = e^{x^2}\sin\left(\frac{2x}{2\pi}\right) = e^{x^2}\sin\left(\frac{x}{\pi}\right)$$

$$\frac{df}{dx} = 2xe^{x^2}\sin\left(\frac{x}{\pi}\right) + \frac{1}{\pi}e^{x^2}\cos\left(\frac{x}{\pi}\right)$$

$\frac{\partial f}{\partial x} = 2xe^{x^2}\sin(2y)$ $\qquad \frac{\partial f}{\partial y} = 2e^{x^2}\cos(2y)$ $\qquad \frac{dy}{dx} = \frac{1}{2\pi}$

$$\frac{df}{dx} = 2xe^{x^2}\sin(2y) + \frac{2}{2\pi}e^{x^2}\cos(2y) \leftarrow$$

$$\frac{df}{dx} = 2xe^{x^2}\sin\left(\frac{x}{\pi}\right) + \frac{1}{\pi}e^{x^2}\cos\left(\frac{x}{\pi}\right) \quad \checkmark$$

Case II. $z = f(x,y)$ $\qquad x = x(t), \; y = y(t)$

$$z = f(x(t), y(t)) = f(t)$$

$\frac{df}{dt}$ should exist.

$$\boxed{\begin{gathered}\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ \dot{f}(t) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\end{gathered}}$$

Ex. $f(x,y) = 2x^2 - 3y^3$ $\qquad$ on unit circle $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$$f(x,y) = f(\cos t, \sin t) = 2\cos^2 t - 3\sin^3 t$$

$$f'(t) = -4\cos t \sin t - 9\sin^2 t \cos t$$

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = -9y^2 \qquad \dot{x} = -\sin t \qquad \dot{y} = \cos t$$

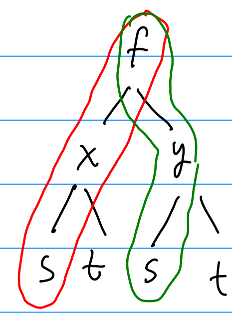
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 4x(-\sin t) + (-9y^2)(\cos t)$$

$$= -4x\sin t - 9y^2\cos t$$

$$= -4\cos t\sin t - 9\sin^2 t\cos t \quad \checkmark$$

Case III. $z = f(x,y) \qquad x = x(s,t) \qquad y = (s,t)$

$$z = f(x,y) = f(x(s,t), y(s,t)) = f(s,t)$$

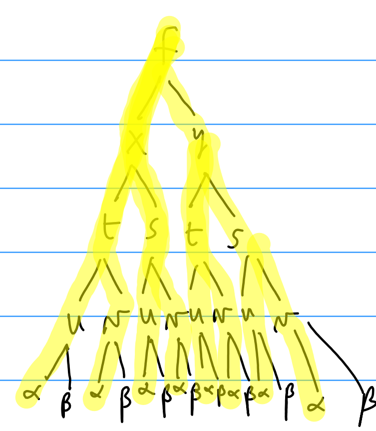


$$\text{and}\begin{cases} \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} \\[2ex] \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} \end{cases}$$

Case IV. $z = f(x,y) \qquad x = x(t,s) \qquad t = t(u,v) \qquad u = u(\alpha,\beta)$

$y = y(t,s) \qquad s = s(u,v) \qquad v = v(\alpha,\beta)$

$z = f(\alpha,\beta)$ Find $\dfrac{\partial z}{\partial \alpha}$?



$$\frac{\partial z}{\partial \alpha} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial u}\frac{\partial u}{\partial \alpha} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial v}\frac{\partial v}{\partial \alpha} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \alpha} +$$

$$+ \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial v}\frac{\partial v}{\partial \alpha} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial u}\frac{\partial u}{\partial \alpha} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \alpha} +$$

$$+ \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial v}\frac{\partial v}{\partial \alpha} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial v}\frac{\partial v}{\partial \alpha}$$

Ex. $f(x,y)$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ Polar coordinates.

Recall the Laplacian: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Find a Polar Equation for $\Delta f = 0$.

$f(r,\theta) = f(x,y)$ where $r = \sqrt{x^2+y^2}$
$\theta = \arctan(y/x)$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}(r,\theta)$ requires a chain rule!

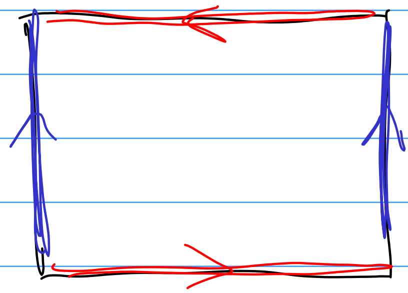
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

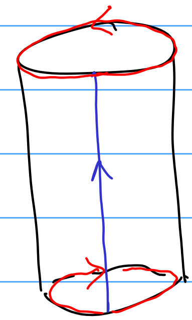
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

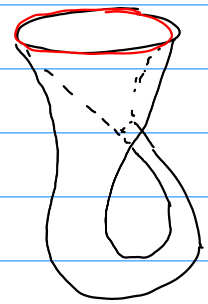
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \left(\frac{\partial^2 f}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

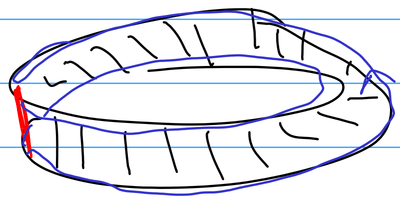
RE. D. it!







Klein Bottle



Möbius