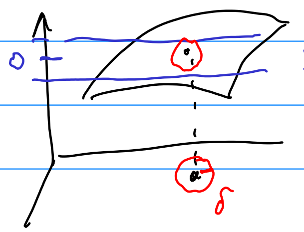
M344 13 Feb 2020

Ex. Prove that $\displaystyle\lim_{\vec{x}\to\vec{0}} \frac{3x^2y}{x^2+y^2}$ exists.

Set $y=0$: $\displaystyle\lim_{x\to 0} \frac{3x^2\cdot 0}{x^2+0^2} = \lim_{x\to 0}\frac{0}{x^2} = 0.$



Set $\varepsilon>0$.
Find $\delta=\delta(\varepsilon)$ to ensure that
whenever $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$,
then $|f(x,y)-L|<\varepsilon$

Proof. Fix $\varepsilon>0$. Put $\delta = \varepsilon/3$.
Then whenever $0<\sqrt{x^2+y^2}<\delta$ we have

$$|f(x)-L| = \left|\frac{3x^2y}{x^2+y^2}-0\right| = \frac{3x^2|y|}{x^2+y^2} \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right)=\varepsilon$$

$$\frac{x^2}{x^2+y^2}\le 1$$

$$|y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$$

Therefore, $\displaystyle\lim_{\vec{x}\to\vec{0}}\frac{3x^2y}{x^2+y^2} = 0.$

Defn. Let f be a function of multiple variables defined in a neighborhood of $\vec{a}$. Then f is continuous at $\vec{a}$ iff

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

or $$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f\left(\lim_{\vec{x}\to\vec{a}}\vec{x}\right) = f(\vec{a})$$

All polynomials, rational functions, trig, exponential, hyperbolic, are continuous on their domains.

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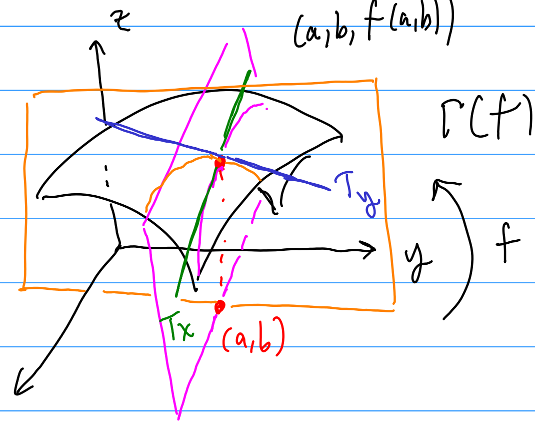
§14.3 Partial Derivatives

Let f be a function of (x,y) and let (a,b) be in the domain.



x-Partial derivative: Set $y=b$, consider the plane.
The partial derivative with respect to x is the slope of T_x that lies in the plane $y=b$.

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

y-derivative: $$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Ex. $f(x,y) = e^{xy}\cos(x+y)$

$$\frac{\partial}{\partial x}\left[f(x,y)\right] = y e^{xy}\cos(x+y) - e^{xy}\sin(x+y)$$

$$\frac{\partial}{\partial y}\left[f(x,y)\right] = x e^{xy}\cos(x+y) - e^{xy}\sin(x+y)$$

Notations:

$$\frac{\partial f}{\partial x} = \partial_x f = D_x f = f_x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

Ex. $u = e^{xy} \sin y$

$\partial_x u = y e^{xy} \sin(y)$ $\qquad\qquad$ $\partial_y u = x e^{xy} \sin y + e^{xy} \cos y$

$\partial_{xx} u = y^2 e^{xy} \sin(y)$

$$\partial_{yx} u = \frac{\partial^2 u}{\partial y \partial x} = \partial_y \partial_x u = e^{xy} \sin y + y\left(x e^{xy} \sin y + e^{xy} \cos y\right)$$
$$= e^{xy} \sin y + yx e^{xy} \sin y + y e^{xy} \cos y$$

$$\partial_{yy} u = x^2 e^{xy} \sin y + x e^{xy} \cos y + x e^{xy} \cos y - e^{xy} \sin y$$

$$\partial_{xy} u = \frac{\partial^2 u}{\partial x \partial y} = e^{xy} \sin y + xy e^{xy} \sin y + y e^{xy} \cos y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Clairaut's Theorem.

Let f be a function of two variables for which all second partials of f are <u>continuous</u>.

Then
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Defn. The <u>Laplacian</u> of a function f of two variables is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

If $\Delta f = 0$, then f is called harmonic.

Ex. $f(x,y) = e^x \sin y$

$\partial_x f = e^x \sin y$ $\qquad\qquad$ $\partial_y f = e^x \cos y$

$\partial_{xx} f = e^x \sin y$ $\qquad\qquad$ $\partial_{yy} f = -e^x \sin y$

$$\Delta f = e^x \sin y - e^x \sin y = 0.$$

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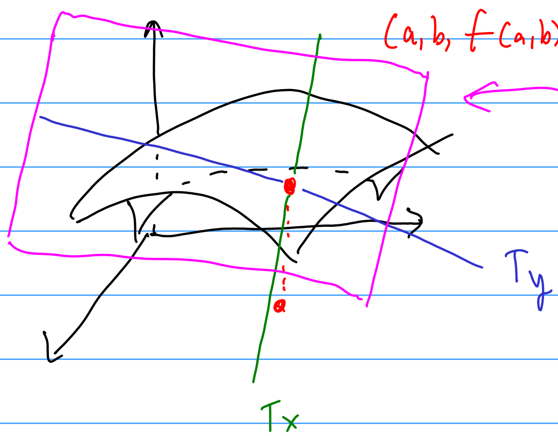
§14.4 Tangent Planes and Linear Approx.



Want an equation: T_x : $y = b$ and passes through $(a, b, f(a,b))$

$$z - z_0 = M(x - x_0)$$
$$z - f(a,b) = \partial_x f(a,b)\,(x-a)$$

T_x: $\quad z = f(a,b) + \partial_x f(a,b)\,(x-a)$

T_y: $x = a$: $\quad z - z_0 = M(y - y_0)$

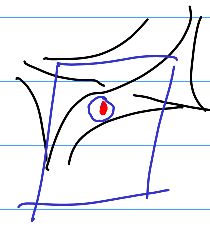
$$z - f(a,b) = \partial_y f(a,b)\,(y-b)$$
$$z = f(a,b) + \partial_y f(a,b)\,(y-b)$$

Def'n. The tangent plane is given by $\boxed{z = f(a,b) + \partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b)}$

In general, $ax+by+cz+d=0$

and $\vec{n}=\langle a,b,c\rangle$

For the tangent plane, the normal vector is $\langle \partial_x f(a,b), \partial_y f(a,b), -1\rangle$.



Ex. $f(x,y)=xe^{xy}$ at $(1,0)$

$f(1,0)=1\cdot e^{1\cdot 0}=1$

$\partial_x f = e^{xy}+xye^{xy}$ $\quad f(1,0)=e^{0}+0=1$

$\partial_y f = x^2e^{xy}$ $\quad f(1,0)=1^2e^{0}=1$

$z=f(1,0)+\partial_x f(1,0)(x-1)+\partial_y(1,0)(y-0)$

$z=1+(x-1)+y$

$\boxed{z=x+y}$

$f(1.1,-0.1)$

$1.1e^{1.1(-0.1)} \approx 1.1-0.1=1$