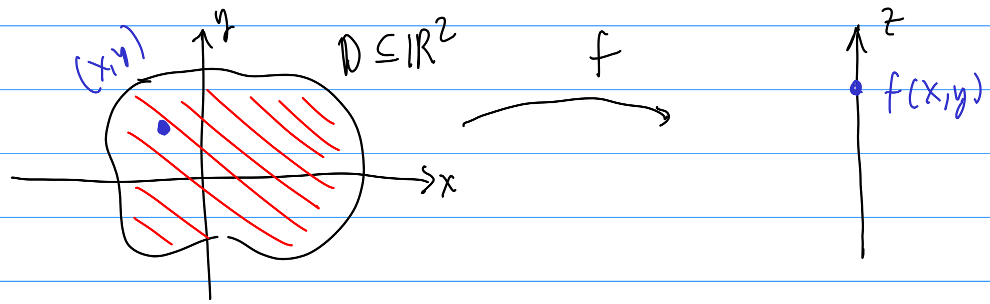
M344 11 Feb 2020

Ch 14 - Partial Derivatives

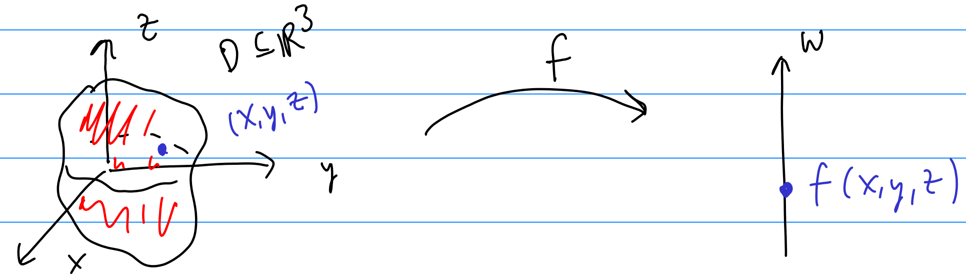
§14.1,2 Functions of multiple variables

Defn. A function f of two variables is function whose domain is a subset of $\mathbb{R}^2$, and whose range is $\mathbb{R}$.



$z = f(x,y)$

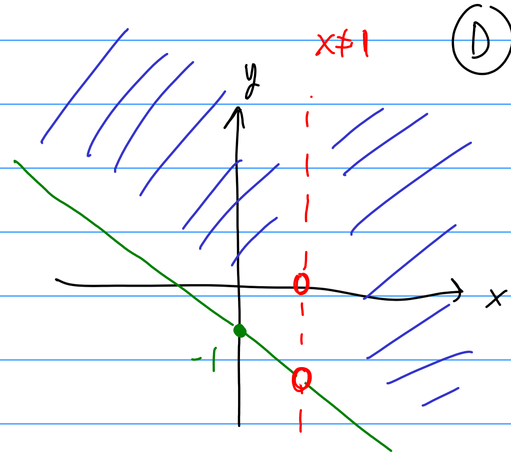
3D :



$w = f(x,y,z)$

In vector notation, we can write $\vec{x} = \langle x, y, z \rangle$ to represent the point in the domain. Then $w = f(\vec{x})$.

Ex. $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ Find the (largest) domain.



Restrictions: $x - 1 \neq 0 \rightarrow x \neq 1$

$x + y + 1 \geq 0$

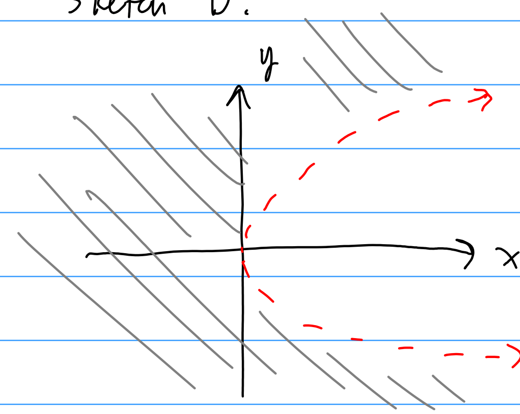
$y \geq -x - 1$

$D = \{ (x,y) \mid x \neq 1 \text{ and } y \geq -x-1 \}$

Ex. $f(x,y) = \ln(y^2 - x)$ Sketch D.

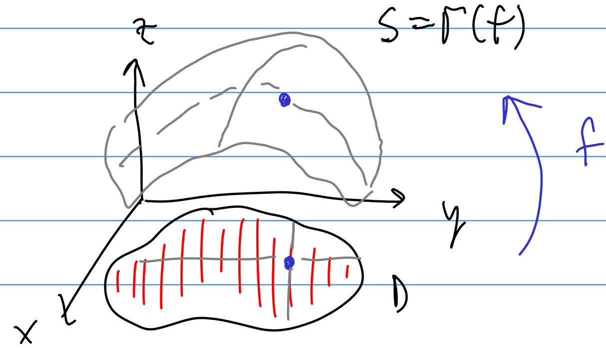
Rest.: $y^2 - x > 0$

$y^2 > x$



<u>Defn.</u> The graph Γ of a function of two variables, f, is the collection of all points satisfying $z = f(x,y)$.

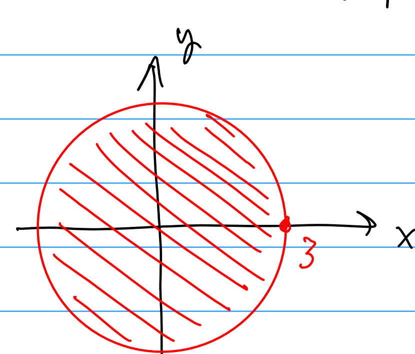
$$\Gamma(f) = \{ (x,y,z) \mid z = f(x,y) \} = \{ (x,y,f(x,y)) \mid (x,y) \in D \}$$



<u>Ex.</u> $g(x,y) = \sqrt{9 - x^2 - y^2}$ Find D, then sketch the graph.

Restrictions: $9 - x^2 - y^2 \geq 0$

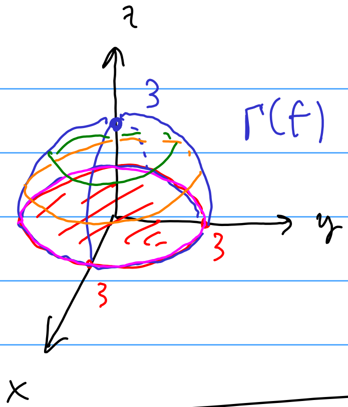
$x^2 + y^2 \leq 9$



$z = g(x,y) \rightarrow z^2 = \sqrt{9 - x^2 - y^2}^{\,2}$ $z \geq 0$

$z^2 = 9 - x^2 - y^2$

$x^2 + y^2 + z^2 = 3^2$ <u>Top half</u> of sphere w/ rad 3.



Ex. $h(x,y) = 2x^2 - 5y^2$ $\qquad$ dom $= \mathbb{R}^2$

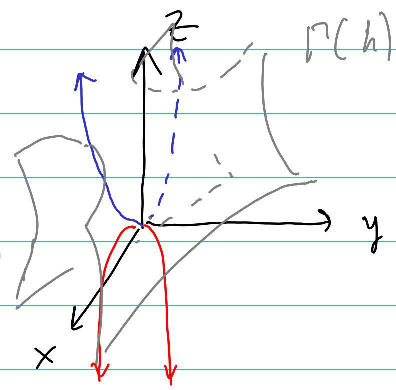
$\downarrow$

$z = 2x^2 - 5y^2$

$y=0$: $z = 2x^2$

$x=0$: $z = -5y^2$

Hyperbolic Paraboloid



Defn. Let $z=f(x,y)$ be a function. The level curves of f are the curves in the xy-plane given by the equations

$f(x,y) = k$ for constant k.

The curves are sometimes called contours.

Ex. $g(x,y) = \sqrt{9-x^2-y^2}$ $\qquad$ dom: $x^2+y^2 \leq 3^2$

range: $[0,3]$

$z = \sqrt{9-x^2-y^2}$

$x^2+y^2 = 9-z^2$

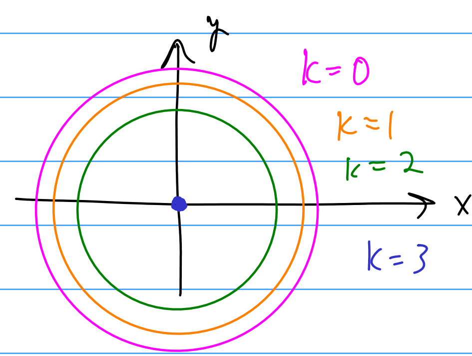
$z=k$,

$k=0$: $x^2+y^2=9$

$k=1$: $x^2+y^2=8$

$k=2$: $x^2+y^2=5$

$k=3$: $x^2+y^2=0$



$z = x^2 + y^2$ $\qquad$ $z = \sqrt{x^2+y^2} \to z^2 = x^2 + y^2$

<u>Ex.</u> $h(x,y) = x^2 + y^2$ $\qquad$ $f(x,y) = \sqrt{x^2+y^2}$

Paraboloid

$z^2 = x^2$

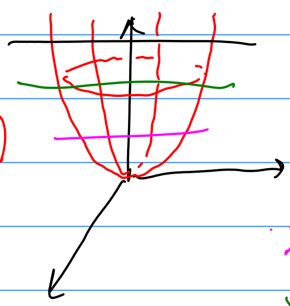
$|z| = |x|$

$z = |x|$

$k=2$

$x^2+y^2=2$

$\Gamma(h)$



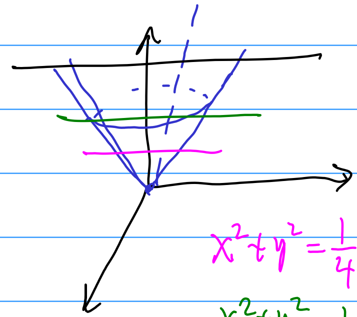
$k=2$

$k=1$

$k=\frac{1}{2}$

$x^2+y^2=\frac{1}{2}$

$x^2+y^2=1$



$k=2$

$x^2+y^2=\frac{1}{4}$

$x^2+y^2=1$

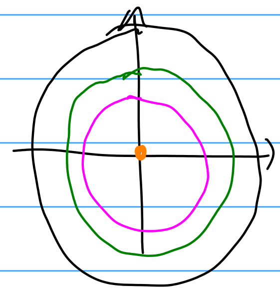
Top nappe of a cone.

$x^2+y^2=4$

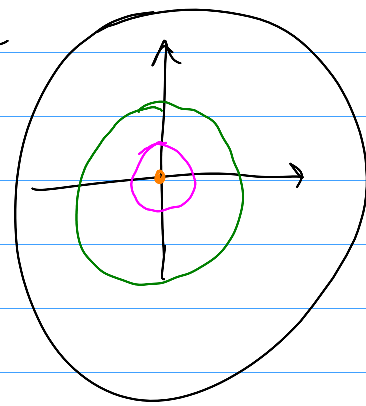
Compare the level curves. $\qquad$ Same domain & range:

$\mathbb{R}^2 \qquad [0,\infty)$

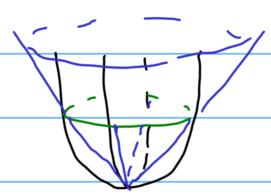
P'loid



Cone



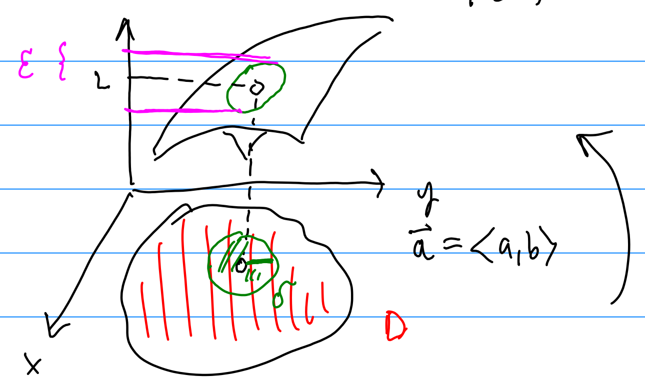
$k=0$



<u>§14.2 Limits</u>

$\Gamma(f)$

<u>Picture:</u>



all $\vec{x}$ such that $0 < \|\vec{x} - \vec{a}\| < \delta$

$|z - L| < \varepsilon$

Defn. Suppose f is a function of two variable that is defined at points arbitrarily close to (a,b), but not necessarily at (a,b). We say the limit of f as (x,y) approaches (a,b) is L provided

whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|z-L| < \varepsilon$ for all $\varepsilon > 0$. $[\delta = \delta(\varepsilon)]$

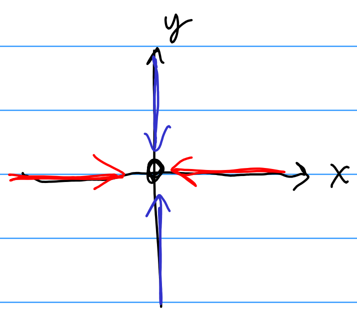
We write $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

RE. Write this out for a function of 3 variables.

Vector notation: $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$

provided, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that whenever $0 < \|\vec{x}-\vec{a}\| < \delta$, then $|z-L| < \varepsilon$.

Ex. $\lim\limits_{\vec{x}\to\vec{0}} \dfrac{x^2-y^2}{x^2+y^2}$ show that this DNE.



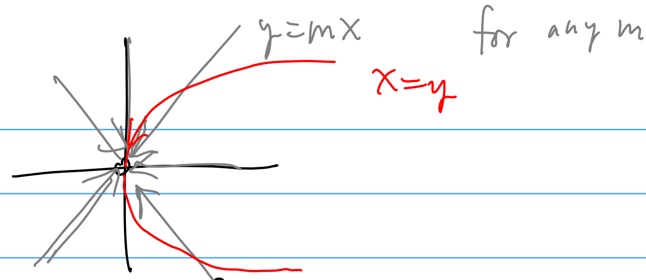
Along $x=0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

$y=0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = +1$

$\neq$

$\lim\limits_{\vec{x}\to\vec{0}} \dfrac{x^2-y^2}{x^2+y^2} = $ DNE

Ex. $\displaystyle\lim_{\vec{x}\to\vec{0}} \frac{xy^2}{x^2+y^4}$



$$\lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{m^2x^3}{m^4x^4+x^2} = 0$$

RE. $x=y^2 \longrightarrow ?$