M344 2/4/20

Chapter 13 Wrap-up

Recall. smooth vector function: $\dot{\vec{r}}$ exists and $\dot{\vec{r}} \neq \vec{0}$.

Unit Tangent vector field: $\vec{T}(t) = \dfrac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}$

Arc length: $s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\, du$

Curvature: $\kappa(s) = \left\| \dfrac{d\vec{T}}{ds} \right\|$ defn ←

$$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$$

Ex. Suppose $y = f(x)$ is a function in the xy plane.
Find a formula for $\kappa(x)$.

$\vec{r}(x) = \langle x, y \rangle = \langle x, f(x) \rangle$ Arc length!

$\dot{\vec{r}}(x) = \vec{r}\,'(x) = \langle 1, f'(x) \rangle \to \|\dot{\vec{r}}(x)\| = \sqrt{1 + (f'(x))^2}$

$$\vec{T}(x) = \frac{\vec{r}\,'(x)}{\|\vec{r}\,'(x)\|} = \left\langle \frac{1}{\sqrt{1+f'(x)^2}}, \frac{f'(x)}{\sqrt{1+f'(x)^2}} \right\rangle$$

$$\vec{T}\,'(x): \quad \left((1+f'(x)^2)^{-1/2} \right)' = -\frac{1}{\cancel{2}}\left(1+f'(x)^2\right)^{-3/2} \cdot \cancel{2} f'(x)\, f''(x)$$

$$= \frac{-f'(x) f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} \leftarrow$$

$$: \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} \right)' = \frac{\cancel{\sqrt{1+f'(x)^2}}^{\,2}\, f''(x) - \dfrac{\overbrace{f'(x)\, \cancel{2} f'(x)}^{f'(x)^2}\, f''(x)}{\cancel{2}\sqrt{1+f'(x)^2}}}{\sqrt{1+f'(x)^2}^{\,2}}$$

$$\sqrt{1+f'(x)^2}^{\,2}$$

$$= \frac{f''(x) + \cancel{f'(x)^2 f''(x)} - \cancel{f'(x)^2 f''(x)}}{\sqrt{1+f'(x)^2}^{\,3}}$$

$$\vec{T}'(x) = \left\langle \frac{-f'(x)f''(x)}{\sqrt{1+f'(x)^2}^{\,3}}, \frac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} \right\rangle$$

$$= \frac{1}{\sqrt{1+f'(x)^2}^{\,3}} \langle -f'(x)f''(x), f''(x) \rangle$$

$$\|\vec{T}'(x)\| = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^{\,\cancel{3}\,2}} \cancel{\sqrt{1+f'(x)^2}}$$

So,

$$\boxed{\kappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^{\,3}}}$$

Plane curve given by a function $y=f(x)$.

Ex. $y=e^x \rightarrow y'=e^x, \; y''=e^x$

$$\kappa(x) = \frac{e^x}{\sqrt{1+e^{2x}}^{\,3}}$$

Thm. Suppose that C is a parametric curve, or $\vec{r}(t)=\langle x(t), y(t)\rangle$.

$$\boxed{\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\sqrt{\dot{x}^2+\dot{y}^2}^{\,3}}}$$

RE. do it.

Ex. $\vec{r}(t) = \left\langle (1-\sin t)\cos t,\ (t+\cos t)\sin t \right\rangle$

Find $\kappa(t)$

$x = \cos t - \sin t \cos t$

$\dot{x} = -\sin t - \sin^2 t + \cos^2 t$

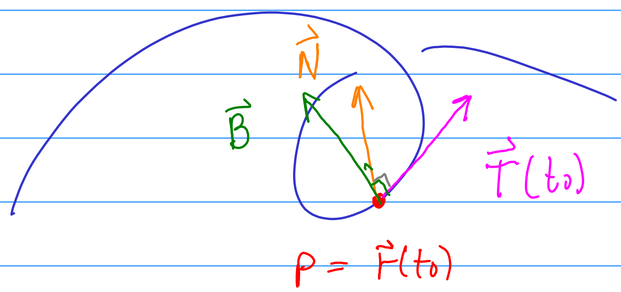
$\ddot{x} = -\cos t - 2\sin t\cos t - 2\cos t \sin t = -\cos t - 4\sin t\cos t$

$y = t\sin t + \cos t \sin t$

$\dot{y} = \sin t + t\cos t + \sin^2 t - \cos^2 t$

$\ddot{y} = \cos t + \cos t - t\sin t - 4\sin t\cos t = 2\cos t - t\sin t - 4\sin t\cos t$

$$\kappa(t) = \frac{\left| -(\sin t + \sin^2 t - \cos^2 t)(2\cos t - t\sin t - 4\sin t\cos t) - (\sin t + t\cos t + \sin^2 t - \cos^2 t)(-\cos t - 4\sin t\cos t) \right|}{\sqrt{(-\sin t - \sin^2 t + \cos^2 t)^2 + (\sin t + t\cos t + \sin^2 t - \cos^2 t)^2}}$$



Frenet Frame

The unit normal vector field is the unit vector that is orthogonal to $\vec{T}$ and points in the direction of curvature.

Thm. $\vec{T} \perp \dot{\vec{T}}$

because we know that if $\|\vec{r}\| = c$, then $\vec{r} \perp \dot{\vec{r}}$, and $\|\vec{T}\| = 1$.

Def'n. The unit normal vector is $\vec{N}(t) = \dfrac{\dot{\vec{T}}(t)}{\|\dot{\vec{T}}(t)\|}$.

The unit binormal vector is $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

Recall $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

Focus, $\|\vec{T}\|, \|\vec{N}\| = 1$ and $\vec{T} \perp \vec{N}$ so $\theta = \pi/2$

so, $\|\vec{B}\| = \|\vec{T}\| \|\vec{N}\| \sin \pi/2 = 1 \cdot 1 \cdot 1 = 1$ ✓

Ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ Find $\vec{T}, \vec{N}, \vec{B}$

$\dot{\vec{r}} = \langle -\sin t, \cos t, 1 \rangle \rightarrow \|\dot{\vec{r}}\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

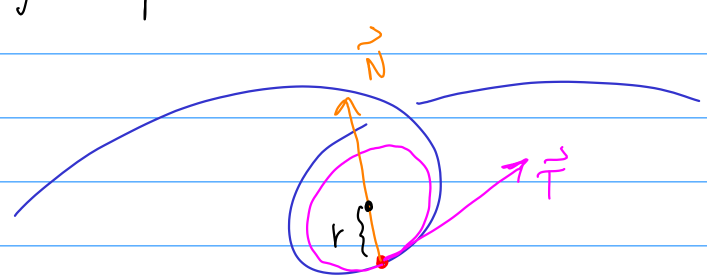
so $\boxed{\vec{T}(t) = \left\langle \frac{-1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \right\rangle}$

$\dot{\vec{T}} = \left\langle \frac{-1}{\sqrt{2}} \cos t, \frac{-1}{\sqrt{2}} \sin t, 0 \right\rangle \rightarrow \|\dot{\vec{T}}\| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \sqrt{1/2} = \frac{1}{\sqrt{2}}$

$\boxed{\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle}$

$$\vec{B}(t) = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1/\sqrt{2} \sin t & \frac{1}{\sqrt{2}} \cos t & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \boxed{\left\langle \frac{1}{\sqrt{2}} \sin t, \frac{-1}{\sqrt{2}} \cos t, 1/\sqrt{2} \right\rangle = \vec{B}(t)}$$

At every point along a space curve we can find a best-fit circle to the curve.



$r = \frac{1}{\kappa}$

The circle • lives in the plane determined by $\vec{T}$ and $\vec{N}$: osculating plane.

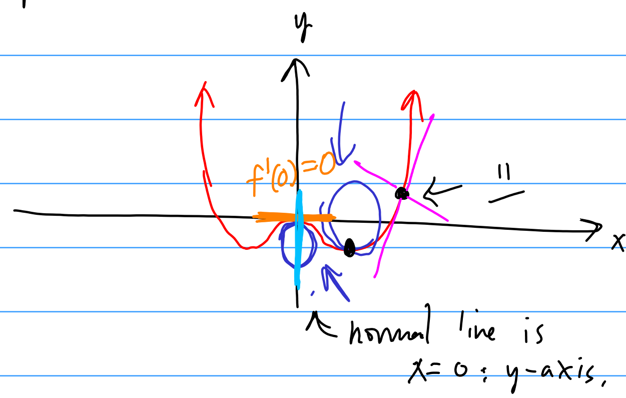
• It is tangent to the curve

• It has the same curvature as the curve, $r = 1/\kappa$

• The center lies on the line determined by $\vec{N}$.

The osculating circle to C at each point.

Ex. $y = x^4 - x^2 = f(x)$



Find osc. circle at $(0,0)$

Need center and radius.

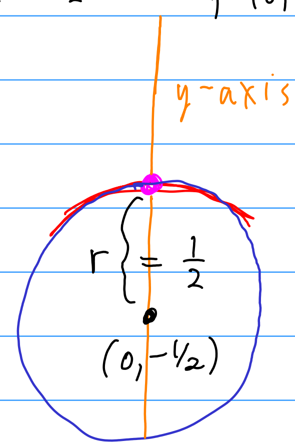
$r = 1/\kappa(0)$

$$\kappa(0) = \frac{|f''(0)|}{\sqrt{1+f'(0)^2}^{\,3}}$$

$f'(x) = 4x^3 - 2x \rightarrow f'(0) = 0$

$f''(x) = 12x^2 - 2 \qquad f''(0) = -2$

$$\kappa(0) = \frac{|-2|}{\sqrt{1+0^2}^{\,3}} = 2$$



$\boxed{r = 1/2}$ center is $(0, -1/2)$.

$$\boxed{x^2 + (y + 1/2)^2 = 1/4}$$