M344 1/30/20

13.2

Thm. Differentiation Rules

Let $\vec{u}, \vec{v}$ be vector functions, f be a function, and c be a scalar.

1. $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \dot{\vec{u}}(t) \pm \dot{\vec{v}}(t)$
2. $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\dot{\vec{u}}(t)$

} Linear

3. $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = \dot{f}(t)\,\vec{u}(t) + f(t)\,\dot{\vec{u}}(t)$
4. $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \dot{\vec{u}}(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\dot{\vec{v}}(t)$
5. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \dot{\vec{u}}(t)\times\vec{v}(t) + \vec{u}(t)\times\dot{\vec{v}}(t)$ order matters!
6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \dot{f}(t)\,\dot{\vec{u}}(f(t))$

RE. Prove some: $\vec{u}(t) = \langle x(t), y(t), z(t)\rangle$ ← use coord def'n to prove the chain rule.

Thm. If $\|\vec{r}(t)\| = C$ for all $t \in \mathrm{dom}(\vec{r})$, then $\vec{r}(t) \perp \dot{\vec{r}}(t)$.

Proof. Case I.) $C = 0$ implies $\vec{r}(t) = \vec{0}$, and $\vec{0}\cdot\dot{\vec{r}} = 0$.

Case II.) $C > 0$

Then $\|\vec{r}(t)\|^2 = C^2$

Then $\|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t) = C^2$

Take $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d}{dt}(C^2)$

$\dot{\vec{r}}\cdot\vec{r} + \vec{r}\cdot\dot{\vec{r}} = 0$

and $2(\dot{\vec{r}}\cdot\vec{r}) = 0$

$\dot{\vec{r}}\cdot\vec{r} = 0$ whence $\vec{r} \perp \dot{\vec{r}}$. ∎

Think. $\|\vec{r}(t)\| = C$

$x^2 + y^2 + z^2 = C^2$

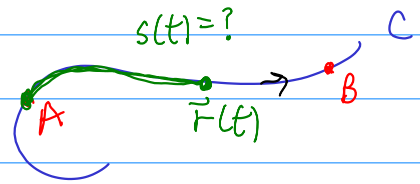
spherical functions

13.3

Back to arc length.

$\vec{r}(t)$ vector function w/ space curve C

The length of the portion of C between $\vec{r}(a)=A$ and $\vec{r}(b)=B$



$$s = \int_a^b \|\dot{\vec{r}}(t)\|\, dt$$

The arc length function starting at $t=a$ and moving in the positive direction is

$$\boxed{s(t) = \int_a^t \|\dot{\vec{r}}(u)\|\, du}$$

Ex. $\dfrac{ds}{dt} = \dfrac{d}{dt}\displaystyle\int_a^t \|\dot{\vec{r}}(u)\|\, du = \|\dot{\vec{r}}(t)\|$

Ex. helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Reparametrize $\vec{r}$ with respect to arc length. $\vec{r}(s)$

$t_0 = 0$

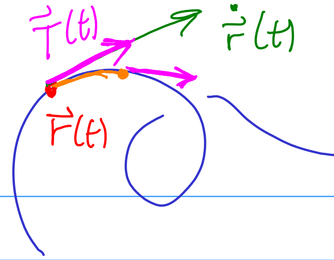
The arc length function is:

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\, du$$

$\dot{\vec{r}}(t) = \langle -\sin t, \cos t, 1 \rangle$ and $\|\dot{\vec{r}}\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, u \Big|_0^t = \sqrt{2}\, t = s \longrightarrow t = \frac{s}{\sqrt{2}}$$

helix: $\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$



$C : \vec{r}(t)$ **smooth**: $\dot{\vec{r}}(t) \neq \vec{0}$

Recall: $\dot{\vec{r}}$ is tangent to C but its length depends on $\vec{r}$.

Defn. The **unit tangent vector field** along C is

$$\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}.$$

K **Defn.** The **curvature** $\varkappa$ of a space curve is the magnitude of the rate of change of the unit tangent vector field with respect to arc length.

$$\varkappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$$ No dots!

Ex. helix: $\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$

$$\boxed{\vec{T}(s) = \frac{d\vec{r}}{ds}} = \left\langle -\frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right\rangle$$

$$\left\| \frac{d\vec{r}}{ds} \right\| = \frac{1}{\sqrt{2}} \sqrt{\sin^2\left(\frac{s}{\sqrt{2}}\right) + \cos^2\left(\frac{s}{\sqrt{2}}\right) + 1^2} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\frac{d\vec{T}}{ds} = \left\langle \frac{-1}{2} \cos\left(\frac{s}{\sqrt{2}}\right), \frac{-1}{2} \sin\left(\frac{s}{\sqrt{2}}\right), 0 \right\rangle$$

$$\varkappa(s) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{1}{2} \sqrt{\cos^2\left(\frac{s}{\sqrt{2}}\right) + \sin^2\left(\frac{s}{\sqrt{2}}\right)} = \frac{1}{2}$$

The helix has constant curvature!

Thm. $\kappa(t) = \dfrac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$

Proof. $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\| = \left\|\dfrac{d\vec{T}/dt}{ds/dt}\right\|$

because: $\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}}{dt}\dfrac{dt}{ds} = \dfrac{d\vec{T}/dt}{ds/dt}$

$$= \left\|\frac{\dot{\vec{T}}}{\|\dot{\vec{r}}\|}\right\|$$

$$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} \quad \blacksquare$$

Ex. (Thm) A circle of radius $a>0$ has constant curvature of $\kappa(t) = 1/a$.

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$\vec{T}(t) = \dfrac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}\|} = \left\langle \dfrac{-a}{a}\sin t, \dfrac{a}{a}\cos t\right\rangle = \langle -\sin t, \cos t\rangle$

$\dot{\vec{T}}(t) = \langle -\cos t, -\sin t\rangle$

and

$\|\dot{\vec{T}}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$

So $\kappa(t) = \dfrac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \dfrac{1}{a} \quad \blacksquare$

Ex. $y = \ln(x)$ $\quad \vec{r}(x) = \langle x, \ln x\rangle \quad x>0$

Compute $\kappa(x)$

$\dot{\vec{r}}(x) = \langle 1, 1/x\rangle$

$\|\dot{\vec{r}}\| = \sqrt{1 + 1/x^2} = \sqrt{\dfrac{x^2+1}{x^2}} = \dfrac{\sqrt{x^2+1}}{x}$

$\vec{T}(x) = \left\langle \dfrac{x}{\sqrt{x^2+1}}, \dfrac{1}{\sqrt{x^2+1}}\right\rangle$

I.

$$\dot{\vec{T}} = \frac{\frac{\sqrt{x^2+1}^{\,2}}{\sqrt{x^2+1}} - \frac{x \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2} = \frac{\cancel{x^2}+1-\cancel{x^2}}{\left(\sqrt{x^2+1}\right)^3}$$

II.

$$\left(x^2+1\right)^{-1/2} \xrightarrow{d/dx} -\frac{1}{\cancel{2}}\left(x^2+1\right)^{-3/2} \cdot \cancel{2}x = \frac{-x}{\left(\sqrt{x^2+1}\right)^3}$$

$$\dot{\vec{T}} = \frac{1}{\left(\sqrt{x^2+1}\right)^3}\langle 1, -x \rangle \qquad \|\dot{\vec{T}}\| = \frac{\sqrt{1+x^2}}{\left(\sqrt{x^2+1}\right)^3} = \frac{1}{\left(\sqrt{x^2+1}\right)^2}$$

So,

$$\kappa(x) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \frac{1}{\left(\sqrt{x^2+1}\right)^2} \cdot \frac{x}{\sqrt{x^2+1}} = \frac{x}{\left(\sqrt{x^2+1}\right)^3}$$